Review Part III
Complex Analysis

Underlined Definitions: May be asked for on exam
Underlined Propositions or Theorems: Proofs may be asked for on exam

Chapter 4.1

Riemann-Stieltjes Integrals

Definition of function of bounded variation and total variation

Proposition (1.3) If $\gamma:[a,b]\to\mathbb{C}$ is piecewise smooth, then γ is of bounded variation and $V(\gamma)=\int_a^b |\gamma'(t)|\,dt$.

Definition of Riemann-Stieltjes Integral

Theorem (1.4) If $f:[a,b]\to\mathbb{C}$ is continuous and if $\gamma:[a,b]\to\mathbb{C}$ is of bounded variation, then the Riemann-Stieltjes integral $\int_a^b f\ d\gamma=\int_a^b f(t)d\gamma(t)$ exists.
(Proof uses Cantor's Theorem II.3.7).

<u>Proposition 1.7</u> Let $f,g:[a,b]\to\mathbb{C}$ be continuous, let $\gamma,\sigma:[a,b]\to\mathbb{C}$ be of bounded variation and let $\alpha,\beta\in\mathbb{C}$. Then,

a) $\int_a^b(\alpha f+\beta g)d\gamma=\alpha\int_a^b f\,d\gamma+\beta\int_a^b g\,d\gamma$

b) $\int_a^b fd(a\gamma+\beta\sigma)=\alpha\int_a^b f\,d\gamma+\beta\int_a^b g\,d\sigma$

Proposition Let $f:[a,b]\to\mathbb{C}$ be continuous and let $\gamma:[a,b]\to\mathbb{C}$ be of bounded variation. If $a<t_0<t_1<\cdots<t_n=b$, then $\int_a^b fd\gamma=\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k} fd\gamma$

<u>Theorem (1.9)</u> If $\gamma:[a,b]\to\mathbb{C}$ is piecewise smooth and $f:[a,b]\to\mathbb{C}$, then $\int_a^b f\ d\gamma=\int_a^b f(t)\gamma'(t)dt$.

Definition for a path $\gamma:[a,b]\to\mathbb{C}$ of trace of γ, $\{\gamma\}$.

Definition of rectifiable path $\gamma:[a,b]\to\mathbb{C}$ and length of $\{\gamma\}=\int_a^b d\gamma$. For γ piece-wise smooth, length of $\{\gamma\}=\int_a^b|\gamma'(t)|\,dt$.

Definition of line integral: Let $\gamma:[a,b]\to\mathbb{C}$ be rectifiable path and let $f:\{\gamma\}\to\mathbb{C}$ be continuous,

define line integral $\int_\gamma f=\int_a^b (f\circ\gamma)d\gamma=\int_a^b f(\gamma(t))d\gamma(t)=\int_\gamma f(z)dz$

Note: if γ piece-wise smooth, then $\int_\gamma f=\int_a^b (f\circ\gamma)d\gamma=\int_a^b f(\gamma(t))d\gamma(t)=\int_a^b f(\gamma(t))\gamma'(t)dt=\int_\gamma f(z)dz$

<u>Problems</u> about computing line integrals using the definition

Definition of a change of parameter φ

Proposition If φ is a change of parameter, i.e., if $\varphi:[c,d]\to[a,b]$, φ is continuous, strictly increasing and φ is onto, then for $\gamma:[a,b]\to\mathbb{C}$ a rectifiable path and $f:\{\gamma\}\to\mathbb{C}$ continuous, then $\int_\gamma f=\int_{\gamma\circ\varphi} f$

Definition: (1) a curve as an equivalance class of rectifiable paths;
(2) the trace of a curve is the trace of a representative;
(3) a curve is smooth if some representative is smooth;
(4) a curve is closed if the initial and terminal points on the trace are the same.

Definition for $\gamma:[a,b]\to\mathbb{C}$ a rectifiable path of $-\gamma$ and of $|\gamma(t)|$ and definition

$$\int_\gamma f(z)|dz|=\int_a^b f(\gamma(t))d|\gamma|(t)$$

<u>Proposition (1.17)</u> Let $\gamma:[a,b]\to\mathbb{C}$ be a rectifiable path and let $f:\{\gamma\}\to\mathbb{C}$ be continuous. Then,

a) $\int_{-\gamma} f=-\int_\gamma f$

b) $|\int_\gamma f|\le\int_\gamma |f||dz|\le\max_{z\in\{\gamma\}}|f(z)|V(\gamma)$

Theorem (Fundamental of Theorem of Calculus for Line Integrals) Let *G* be a region and let γ be a rectifiable path in *G* with initial and terminal points α and β, resp. If $f:G\to\mathbb{C}$ is continuous and if *f* has a primitive on *G*, say *F*, then $\int_\gamma f=F(z)\Big|_\alpha^\beta$.

Corollary Let *G* be a region and let γ be a closed rectifiable path in G. If $f:G\to\mathbb{C}$ is continuous and if *f* has a primitive on *G*, say *F*, then $\int_\gamma f=0$.

<u>Problems</u> about computing line integrals using the Fund. Thm. of Calc. for Line Integrals

Chapter 4.2

Proposition 2.1 (Leibnitz's Rule)

<u>Integrals</u>

a) $\displaystyle\int_{|w|=1} (w-z)^n dw = 0, \;\; n = 0,1,2,3,\cdots$

b) $\displaystyle\int_{|w|=1} \frac{dw}{(w-z)^n} = 0, \;\; \begin{cases} n = 2,3,4,5,\cdots \\ |z| \neq 1 \end{cases}$

c) $\displaystyle\int_{|w|=1} \frac{dw}{w-z} = \begin{cases} 0, & |z|>1 \\ 2\pi i, & |z|<1 \end{cases}$

<u>Cauchy Integral Formula #0</u> Let $f : G \to \mathbb{C}$ be analytic and suppose that $\overline{B(a,r)} \subset G$. For $z \in B(a,r)$, $\displaystyle f(z) = \frac{1}{2\pi i} \int_{|w-a|=r} \frac{f(w)}{w-z}\, dw$

<u>Problems</u> about computing line integrals using the CIF #0

<u>Lemma (2.7)</u> Let γ be a rectifiable curve. Suppose that F_n and F are continuous on $\{\gamma\}$ and that $\{F_n\}$ converges uniformly on $\{\gamma\}$ to F. Then, $\displaystyle\lim_{n\to\infty} \int_{\gamma} F_n = \int_{\gamma} \lim_{n\to\infty} F_n = \int_{\gamma} F$

<u>Theorem 2.8</u> Let G be a region and let $f : G \to \mathbb{C}$ be analytic. Let $B(a,R) \subset G$. Then, f has a power series representation on $B(a,R)$, say

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \tag{1}$$

where the coefficients $\displaystyle a_n = \frac{f^{(n)}(a)}{n!} = \frac{1}{2\pi i} \int_{|w-a|=\rho} \frac{f(w)}{(w-a)^{n+1}}\, dw$, for any choice $0 < \rho < R$. Furthermore, the radius of convergence of the power series (1) is at least R.

Corollaries (Hypothesis: Let G be a region and let $f : G \to \mathbb{C}$ be analytic. Let $B(a,R) \subset G$.)

a) the radius of convergence of the power series (1) is equal to $\text{dist}(a, \partial G)$, i.e., the distance (from a) to the nearest singularity of f

b) $f^{(n)}(a) = \dfrac{n!}{2\pi i} \displaystyle\int_{|w-a|=\rho} \frac{f(w)}{(w-a)^{n+1}}\, dw$

c) Cauchy's Estimate If $|f(z)| \le M$ on $B(a,R)$, then $|f^{(n)}(a)| \le \dfrac{n!M}{R^n}$.

d) f has a primitive on $B(a,R)$, namely $F(z) = \displaystyle\sum_{n=0}^{\infty} \frac{a_n}{n+1}(z-a)^{n+1}$

e) Proposition 2.15 Suppose γ is a closed rectifiable curve in $B(a,R)$. Then, $\displaystyle\int_\gamma f = 0$

Chapter 4.3

Division Algorthim

Definition: Let G be a region and let $f : G \to \mathbb{C}$ be analytic and let $f(a) = 0$. We say that f has a zero of order m (multiplicity m) at $z = a$ if

a) there exists $g \in A(G)$ such that (i) $f(z) = (z-a)^m g(z)$ and (ii) $g(a) \ne 0$

or alternatively

b) (i) $f(a) = f'(a) = f''(a) = \cdots = f^{(m-1)}(a) = 0$ and $f^{(m)}(a) \ne 0$

Definition: entire function

Liouville's Theorem If f is a bounded entire function, then f is constant.

Fundamental Theorem of Algebra Every non-constant polynomial with complex coefficients has a root in $\mathbb{C}$

Corollary: Every polynomial with complex coefficients of degree n has exactly n roots (counted according to multiplicity)

Identity Theorem: Let G be a region. Let f be analytic on G. TFAE

a) $f \equiv 0$

b) there exists a point $a \in G$ such that $f^{(n)}(a) = 0, n = 0, 1, 2, 3, \cdots$

c) $Z_f = \{z \in G : f(z) = 0\}$ has a limit point in G.

Corollaries:

a) Let $g, h \in A(G)$ and $g(z) = h(z)$ for $z \in S \subset G$. If S has a limit point in G, then $g \equiv h$.

b) Let G be a region. Let f be analytic on G. Suppose that f is not identically 0 on G. If $a \in G$ and $f(a) = 0$, then there exists an integer m such that f has a zero at $z = a$ of multiplicity m.

c) Isolated Zeros. Let G be a region. Let f be analytic on G. Suppose that f is not identically 0 on G. If $a \in G$ and $f(a) = 0$, then there exists an $R > 0$ such that on $B(a, R) \setminus \{a\}$ we have $f(z) \neq 0$.

d) Let G be a region. Let f be analytic on G. Suppose that f is not identically 0 on G. Let K be a compact subset of G. Then, f has at most a finite number of zeros on K. Further, f has at most a countable number of zeros on G.

Maximum Modulus Theorem Let G be a region. Let f be analytic on G. If there exists a point $a \in G$ such that $|f(a)| \geq |f(z)|$ for all $z \in G$, then f is constant on G.

Extension Let G be a region. Let f be analytic on G. If there exists a point $a \in G$ and $r > 0$ such that $|f(a)| \geq |f(z)|$ for all $z \in B(a, r)$, then f is constant on G.

Chapter 4.4

Proposition: Let γ be a closed rectifiable curve and let $a \notin \{\gamma\}$. Then, $\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a}$ is an integer.

Definition: Winding number of γ wrt to a (index of γ wrt to a) $n(\gamma, a) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a}$, for γ a closed rectifiable curve and $a \notin \{\gamma\}$.

Interpretation: $2\pi\ n(\gamma, a)$ represents the total change in $\arg(\gamma(t) - a)$ as $\gamma(t)$ parametrizes the curve $\{\gamma\}$, i.e., $n(\gamma, a)$ represents the total number of times γ winds around a.

Theorem Let γ be a closed rectifiable curve. Then, $n(\gamma, a)$ is constant on components of $\mathbb{C} \setminus \{\gamma\}$. Also, $n(\gamma, a) = 0$ on the unbounded component of $\mathbb{C} \setminus \{\gamma\}$

Cauchy's Integral Formula (#0). Let G be a region in $\mathbb{C}$, let $\overline{B(a,r)} \subset G$ and let γ be the circle $C(a,r)$, oriented positively. Let $f \in A(G)$. Then, for $z \in B(a,r)$, $f(z) = \dfrac{1}{2\pi i} \displaystyle\int_\gamma \frac{f(w)}{w-z}\, dw$.

Cauchy's Theorem (#0). Let G be a region in $\mathbb{C}$, let $B(a,r) \subset G$ and let γ be a closed rectifiable curve in $B(a,r)$. Let $f \in A(G)$. Then, $\displaystyle\int_\gamma f = 0$.

Chapter 4.5

Definition. $\gamma \approx 0$ for a closed rectifiable curve γ lying in a region G if . . .

Lemma 5.1. Let γ be a rectifiable curve and let φ be continuous on $\{\gamma\}$. For each $m \geq 1$, let $f_m(z) = \displaystyle\int_\gamma \frac{\varphi(w)}{(w-z)^m}\, dw$ for $z \in \mathbb{C} \setminus \{\gamma\}$. Then, $f_m \in A(\mathbb{C} \setminus \{\gamma\})$ and $f_m'(z) = m\, f_{m+1}(z)$.

Cauchy's Integral Formula (#1). Let G be a region in $\mathbb{C}$ and let $f \in A(G)$. Let γ be a closed rectifiable curve in G such that $\gamma \approx 0$. Then, for $z \in G \setminus \{\gamma\}$, $n(\gamma; z) f(z) = \dfrac{1}{2\pi i} \displaystyle\int_\gamma \frac{f(w)}{w-z}\, dw$.

Cauchy's Integral Formula (#2). Let G be a region in $\mathbb{C}$ and let $f \in A(G)$. Let $\gamma_1, \gamma_2, \cdots, \gamma_m$ be closed rectifiable curves in G such that $\gamma_1 + \gamma_2 + \cdots + \gamma_m \approx 0$. Then, for $z \in G \setminus \bigcup_{k=1}^n \{\gamma_k\}$,
$\displaystyle\sum_{k=1}^m n(\gamma_k; z) f(z) = \sum_{k=1}^m \frac{1}{2\pi i} \int_{\gamma_k} \frac{f(w)}{w-z}\, dw$.

Cauchy's Theorem (#1). Let G be a region in $\mathbb{C}$ and let $f \in A(G)$. Let γ be a closed rectifiable curve in G such that $\gamma \approx 0$. Then, $\displaystyle\int_\gamma f = 0$

Cauchy's Theorem (#2). Let G be a region in $\mathbb{C}$ and let $f \in A(G)$. Let $\gamma_1, \gamma_2, \cdots, \gamma_m$ be a closed rectifiable curves in G such that $\gamma_1 + \gamma_2 + \cdots + \gamma_m \approx 0$. Then, $\displaystyle\sum_{k=1}^m \int_{\gamma_k} f = 0$

<u>Corollary.</u> Let G be a region in $\mathbb{C}$ and let $f \in A(G)$. Let γ be a closed rectifiable curve in G such that $\gamma \approx 0$. Then, for $z \in G \setminus \{\gamma\}$, $n(\gamma; z) f^{(k)}(z) = \dfrac{k!}{2\pi i} \displaystyle\int_\gamma \frac{f(w)}{(w-z)^{k+1}}\, dw$.

<u>Morera's Theorem</u> Let G be a region in $\mathbb{C}$ and let $f \in C(G)$. Suppose that $\displaystyle\int_T f = 0$ for every triangular path $T \subset G$. Then, $f \in A(G)$.