Chapter 4.1

Riemann-Stieltjes Integrals

Definition of function of bounded variation and total variation

Proposition (1.3) If \( \gamma : [a, b] \rightarrow \mathbb{C} \) is piecewise smooth, then \( \gamma \) is of bounded variation and
\[
V(\gamma) = \int_a^b |\gamma'(t)| \, dt .
\]

Definition of Riemann-Stieltjes Integral

Theorem (1.4) If \( f : [a, b] \rightarrow \mathbb{C} \) is continuous and if \( \gamma : [a, b] \rightarrow \mathbb{C} \) is of bounded variation, then the Riemann-Stieltjes integral
\[
\int_a^b f(t) \, d\gamma(t)
\]
exists.

(Proof uses Cantor’s Theorem II.3.7).

Proposition 1.7 Let \( f, g : [a, b] \rightarrow \mathbb{C} \) be continuous, let \( \gamma, \sigma : [a, b] \rightarrow \mathbb{C} \) be of bounded variation and let \( \alpha, \beta \in \mathbb{C} . \) Then,
\[
\begin{align*}
a) \int_a^b (\alpha f + \beta g) \, d\gamma &= \alpha \int_a^b f \, d\gamma + \beta \int_a^b g \, d\gamma \\
b) \int_a^b f \, a\gamma + \beta \sigma &= \alpha \int_a^b f \, d\gamma + \beta \int_a^b g \, d\sigma
\end{align*}
\]

Proposition Let \( f : [a, b] \rightarrow \mathbb{C} \) be continuous and let \( \gamma : [a, b] \rightarrow \mathbb{C} \) be of bounded variation. If
\[
a < t_0 < t_1 < \cdots < t_n = b ,
\]
then
\[
\int_a^b f \, d\gamma = \int_{t_0}^{t_1} f \, d\gamma + \sum_{k=1}^{n-1} \int_{t_k}^{t_{k+1}} f \, d\gamma
\]

Theorem (1.9) If \( \gamma : [a, b] \rightarrow \mathbb{C} \) is piecewise smooth and \( f : [a, b] \rightarrow \mathbb{C} \), then
\[
\int_a^b f(\gamma) \, d\gamma = \int_a^b f(t) \gamma'(t) \, dt .
\]

Definition for a path \( \gamma : [a, b] \rightarrow \mathbb{C} \) of trace of \( \gamma \), \( \{\gamma\} \).

Definition of rectifiable path \( \gamma : [a, b] \rightarrow \mathbb{C} \) and length of \( \{\gamma\} = \int_a^b d\gamma . \) For \( \gamma \) piece-wise smooth, length of \( \{\gamma\} = \int_a^b |\gamma'(t)| \, dt . \)
Definition of line integral: Let \( \gamma: [a, b] \to \mathbb{C} \) be a rectifiable path and let \( f: \{ \gamma \} \to \mathbb{C} \) be continuous, define line integral 
\[
\int_{\gamma} f = \int_{a}^{b} (f \circ \gamma) d\gamma = \int_{a}^{b} f(\gamma(t)) d\gamma(t) = \int_{a}^{b} f(z) dz
\]

Note: if \( \gamma \) piece-wise smooth, then 
\[
\int_{\gamma} f = \int_{a}^{b} (f \circ \gamma) d\gamma = \int_{a}^{b} f(\gamma(t)) d\gamma(t) = \int_{a}^{b} f(\gamma(t)) \gamma'(t) dt = \int_{a}^{b} f(z) dz
\]

Problems about computing line integrals using the definition

Definition of a change of parameter \( \varphi \)

Proposition If \( \varphi \) is a change of parameter, i.e., if \( \varphi: [c, d] \to [a, b] \), \( \varphi \) is continuous, strictly increasing and \( \varphi \) is onto, then for \( \gamma: [a, b] \to \mathbb{C} \) a rectifiable path and \( f: \{ \gamma \} \to \mathbb{C} \) continuous, then 
\[
\int_{\gamma} f = \int_{\gamma \circ \varphi} f
\]

Definition: (1) a curve as an equivalence class of rectifiable paths;  
(2) the trace of a curve is the trace of a representative;  
(3) a curve is smooth if some representative is smooth;  
(4) a curve is closed if the initial and terminal points on the trace are the same.

Definition for \( \gamma: [a, b] \to \mathbb{C} \) a rectifiable path of \(-\gamma\) and of \(\mid \gamma(t)\mid\) and definition
\[
\int_{\gamma} f(z) |dz| = \int_{a}^{b} f(\gamma(t)) |\gamma'(t)|
\]

Proposition (1.17) Let \( \gamma: [a, b] \to \mathbb{C} \) be a rectifiable path and let \( f: \{ \gamma \} \to \mathbb{C} \) be continuous. Then,

a) \[
\int_{-\gamma} f = -\int_{\gamma} f
\]

b) \[
\left| \int_{\gamma} f \right| \leq \int_{\gamma} |f| |dz| \leq \max_{z \in [\gamma]} |f(z)| V(\gamma)
\]

Theorem (Fundamental of Theorem of Calculus for Line Integrals) Let \( G \) be a region and let \( \gamma \) be a rectifiable path in \( G \) with initial and terminal points \( \alpha \) and \( \beta \), resp. If \( f: G \to \mathbb{C} \) is continuous and if \( f \) has a primitive on \( G \), say \( F \), then 
\[
\int_{\gamma} f = F(z)|_{\alpha}^{\beta}
\]

Corollary Let \( G \) be a region and let \( \gamma \) be a closed rectifiable path in \( G \). If \( f: G \to \mathbb{C} \) is continuous and if \( f \) has a primitive on \( G \), say \( F \), then 
\[
\int_{\gamma} f = 0
\]

Problems about computing line integrals using the Fund. Thm. of Calc. for Line Integrals
Chapter 4.2

Proposition 2.1 (Leibnitz's Rule)

Integrals

a) \[ \int_{|w|=1} (w-z)^n \, dw = 0, \quad n = 0, 1, 2, 3, \ldots \]

b) \[ \int_{|w|=1} \frac{dw}{(w-z)^n} = 0, \quad \left\{ \begin{array}{l} n = 2, 3, 4, 5, \ldots \\ |z| \neq 1 \end{array} \right. \]

c) \[ \int_{|w|=1} \frac{dw}{w-z} = \left\{ \begin{array}{l} 0, \quad |z| > 1 \\ 2\pi i, \quad |z| < 1 \end{array} \right. \]

Cauchy Integral Formula #0 Let \( f : G \to \mathbb{C} \) be analytic and suppose that \( B(a, r) \subset G \). For \( z \in B(a, r) \),
\[ f(z) = \frac{1}{2\pi i} \int_{|w-a|=r} \frac{f(w)}{w-z} \, dw \]

Problems about computing line integrals using the CIF #0

Lemma (2.7) Let \( \gamma \) be a rectifiable curve. Suppose that \( F_n \) and \( F \) are continuous on \( \{\gamma\} \) and that \( \{F_n\} \) converges uniformly on \( \{\gamma\} \) to \( F \). Then,
\[ \lim_{n \to \infty} \int_{\gamma} F_n = \int_{\gamma} F \]

Theorem 2.8 Let \( G \) be a region and let \( f : G \to \mathbb{C} \) be analytic. Let \( B(a, R) \subset G \). Then, \( f \) has a power
series representation on \( B(a, R) \), say
\[ f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \quad (1) \]

where the coefficients \( a_n = \frac{f^{(n)}(a)}{n!} = \frac{1}{2\pi i} \int_{|w-a|=\rho} \frac{f(w)}{(w-a)^{n+1}} \, dw \), for any choice \( 0 < \rho < R \). Furthermore, the radius of convergence of the power series \( (1) \) is at least \( R \).
Corollaries (Hypothesis: Let $G$ be a region and let $f : G \to \mathbb{C}$ be analytic. Let $B(a, R) \subset G$.)

a) the radius of convergence of the power series (1) is equal to $\text{dist}(a, \partial G)$, i.e., the distance (from $a$) to the nearest singularity of $f$

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_{|w-a|=\rho} \frac{f(w)}{(w-a)^{n+1}} \, dw$$

c) **Cauchy’s Estimate** If $|f(z)| \leq M$ on $B(a, R)$, then $|f^{(n)}(a)| \leq \frac{n!M}{R^n}$.

d) $f$ has a primitive on $B(a, R)$, namely $F(z) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (z-a)^{n+1}$

e) Proposition 2.15 Suppose $\gamma$ is a closed rectifiable curve in $B(a, R)$. Then, $\int_{\gamma} f = 0$

**Chapter 4.3**

**Division Algorithm**

**Definition:** Let $G$ be a region and let $f : G \to \mathbb{C}$ be analytic and let $f(a) = 0$. We say that $f$ has a zero of order $m$ (multiplicity $m$) at $z = a$ if

a) there exists $g \in \mathcal{A}(G)$ such that (i) $f(z) = (z-a)^m g(z)$ and (ii) $g(a) \neq 0$

or alternatively

b) (i) $f(a) = f'(a) = f''(a) = \cdots = f^{(m-1)}(a) = 0$ and $f^{(m)}(a) \neq 0$

**Definition:** entire function

**Liouville’s Theorem** If $f$ is a bounded entire function, then $f$ is constant.

**Fundamental Theorem of Algebra** Every non-constant polynomial with complex coefficients has a root in $\mathbb{C}$

**Corollary:** Every polynomial with complex coefficients of degree $n$ has exactly $n$ roots (counted according to multiplicity)
Identity Theorem: Let $G$ be a region. Let $f$ be analytic on $G$. TFAE

a) $f \equiv 0$

b) there exists a point $a \in G$ such that $f^{(n)}(a) = 0, n = 0, 1, 2, 3, \ldots$

c) $Z_f = \{z \in G : f(z) = 0\}$ has a limit point in $G$.

Corollaries:

a) Let $g, h \in A(G)$ and $g(z) = h(z)$ for $z \in S \subset G$. If $S$ has a limit point in $G$, then $g \equiv h$.

b) Let $G$ be a region. Let $f$ be analytic on $G$. Suppose that $f$ is not identically 0 on $G$. If $a \in G$ and $f(a) = 0$, then there exists an integer $m$ such that $f$ has a zero at $z = a$ of multiplicity $m$.

c) Isolated Zeros. Let $G$ be a region. Let $f$ be analytic on $G$. Suppose that $f$ is not identically 0 on $G$. If $a \in G$ and $f(a) = 0$, then there exists an $R > 0$ such that on $B(a, R) \setminus \{a\}$ we have $f(z) \neq 0$.

d) Let $G$ be a region. Let $f$ be analytic on $G$. Suppose that $f$ is not identically 0 on $G$. Let $K$ be a compact subset of $G$. Then, $f$ has at most a finite number of zeros on $K$. Further, $f$ has at most a countable number of zeros on $G$.

Maximum Modulus Theorem: Let $G$ be a region. Let $f$ be analytic on $G$. If there exists a point $a \in G$ such that $|f(a)| \geq |f(z)|$ for all $z \in G$, then $f$ is constant on $G$.

Extension: Let $G$ be a region. Let $f$ be analytic on $G$. If there exists a point $a \in G$ and $r > 0$ such that $|f(a)| \geq |f(z)|$ for all $z \in B(a, r)$, then $f$ is constant on $G$.

Chapter 4.4

Proposition: Let $\gamma$ be a closed rectifiable curve and let $a \notin \{\gamma\}$. Then, $\frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z - a}$ is an integer.

Definition: Winding number of $\gamma$ wrt to $a$ (index of $\gamma$ wrt to $a$) $n(\gamma, a) = \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z - a}$, for $\gamma$ a closed rectifiable curve and $a \notin \{\gamma\}$.

Interpretation: $2\pi n(\gamma, a)$ represents the total change in $\arg(\gamma(t) - a)$ as $\gamma(t)$ parametrizes the curve $\{\gamma\}$, i.e., $n(\gamma, a)$ represents the total number of times $\gamma$ winds around $a$.

Theorem: Let $\gamma$ be a closed rectifiable curve. Then, $n(\gamma, a)$ is constant on components of $\mathbb{C} \setminus \{\gamma\}$. Also, $n(\gamma, a) = 0$ on the unbounded component of $\mathbb{C} \setminus \{\gamma\}$.
Cauchy’s Integral Formula (#0). Let $G$ be a region in $\mathbb{C}$, let $\overline{B(a,r)} \subset G$ and let $\gamma$ be the circle $C(a,r)$, oriented positively. Let $f \in A(G)$. Then, for $z \in B(a,r)$, 
\[ f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w-z} \, dw. \]

Cauchy’s Theorem (#0). Let $G$ be a region in $\mathbb{C}$, let $B(a,r) \subset G$ and let $\gamma$ be a closed rectifiable curve in $B(a,r)$. Let $f \in A(G)$. Then, $\int_{\gamma} f = 0$.

Chapter 4.5

Definition. $\gamma \approx 0$ for a closed rectifiable curve $\gamma$ lying in a region $G$ if . . .

Lemma 5.1. Let $\gamma$ be a rectifiable curve and let $\phi$ be continuous on $\{\gamma\}$. For each $m \geq 1$, let 
\[ f_m(z) = \frac{\phi(w)}{\gamma(w-z)^m} \, dw \quad \text{for} \quad z \in \mathbb{C} \setminus \{\gamma\}. \]
Then, $f_m \in A(\mathbb{C} \setminus \{\gamma\})$ and $f_m(z) = m f_{m+1}(z)$.

Cauchy’s Integral Formula (#1). Let $G$ be a region in $\mathbb{C}$ and let $f \in A(G)$. Let $\gamma$ be a closed rectifiable curve in $G$ such that $\gamma \approx 0$. Then, for $z \in G \setminus \{\gamma\}$, 
\[ n(\gamma;z)f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w-z} \, dw. \]

Cauchy’s Integral Formula (#2). Let $G$ be a region in $\mathbb{C}$ and let $f \in A(G)$. Let $\gamma_1, \gamma_2, \ldots, \gamma_m$ be closed rectifiable curves in $G$ such that $\gamma_1 + \gamma_2 + \ldots + \gamma_m \approx 0$. Then, for $z \in G \setminus \bigcup_{k=1}^{m} \{\gamma_k\}$, 
\[ \sum_{k=1}^{m} n(\gamma_k;z)f(z) = \sum_{k=1}^{m} \frac{1}{2\pi i} \int_{\gamma_k} \frac{f(w)}{w-z} \, dw. \]

Cauchy’s Theorem (#1). Let $G$ be a region in $\mathbb{C}$ and let $f \in A(G)$. Let $\gamma$ be a closed rectifiable curve in $G$ such that $\gamma \approx 0$. Then, $\int_{\gamma} f = 0$.

Cauchy’s Theorem (#2). Let $G$ be a region in $\mathbb{C}$ and let $f \in A(G)$. Let $\gamma_1, \gamma_2, \ldots, \gamma_m$ be a closed rectifiable curves in $G$ such that $\gamma_1 + \gamma_2 + \ldots + \gamma_m \approx 0$. Then, 
\[ \sum_{k=1}^{m} \int_{\gamma_k} f = 0. \]

Corollary. Let $G$ be a region in $\mathbb{C}$ and let $f \in A(G)$. Let $\gamma$ be a closed rectifiable curve in $G$ such that $\gamma \approx 0$. Then, for $z \in G \setminus \{\gamma\}$, 
\[ n(\gamma;z)f^{(k)}(z) = \frac{k!}{2\pi i} \int_{\gamma} \frac{f(w)}{(w-z)^{k+1}} \, dw. \]

Morera’s Theorem. Let $G$ be a region in $\mathbb{C}$ and let $f \in C(G)$. Suppose that $\int_{T} f = 0$ for every triangular path $T \subset G$. Then, $f \in A(G)$. 

Morera’s Theorem. Let $G$ be a region in $\mathbb{C}$ and let $f \in C(G)$. Suppose that $\int_{T} f = 0$ for every triangular path $T \subset G$. Then, $f \in A(G)$. 

