Pg 20

1. $\Rightarrow$ let x be limit point of A $\quad \exists \{x_n\} \subset A \ni x_n \to x$

by Prop 3.2 $\quad x \in A$

$\Leftarrow$ let $\{x_n\} \subset A \ni x_n \to x \quad$ Wolog $\quad x \notin \{x_n\}$

For $k=1 \quad \exists \; x_{n_1} \in \{x_n\} \ni x_{n_1} \in B(x,1)$

For $k=2 \quad \exists \; x_{n_2} \in \{x_n\} \ni x_{n_2} \in B(x,\frac{1}{2}) \ni d(x,x_{n_2}) < d(x,x_{n_1})$

For $k=3 \quad \exists \; x_{n_3} \in \{x_n\} \ni x_{n_3} \in B(x,\frac{1}{3}) \ni d(x,x_{n_3}) < d(x,x_{n_2})$

$\vdots$

$\{x_{n_k}\}$ is distinct subsequence of $\{x_n\}$ & $\{x_{n_k}\} \to x$

$\Rightarrow x$ is lim pt of $A \Rightarrow x \in A$

3. Since $A \subset \bar{A} \Rightarrow \text{diam}(A) \leq \text{diam}(\bar{A})$

Let $\varepsilon > 0 \quad \exists \; x, y \in \bar{A} \ni \text{diam}(\bar{A}) - \frac{\varepsilon}{3} < d(x,y)$

$\exists \; \hat{x} \in A \ni d(x,\hat{x}) < \frac{\varepsilon}{3}$

$\exists \; \hat{y} \in A \ni d(y,\hat{y}) < \frac{\varepsilon}{3}$

$\therefore \; \text{diam}(\bar{A}) - \frac{\varepsilon}{3} < d(\hat{x},x) + d(\hat{x},\hat{y}) + d(\hat{y},y)$

$\text{diam}(\bar{A}) \leq d(\hat{x},\hat{y}) + \varepsilon \leq \text{diam}(A) + \varepsilon$

$\Rightarrow \text{diam}(\bar{A}) \leq \text{diam}(A)$

5 Let $\{x_n\} \to x \quad \forall \varepsilon \; \exists N \ni d(x_n,x) < \frac{\varepsilon}{2}$ when $n \geq N$

$\Rightarrow n,m \geq N$ we have $d(x_n,x_m) \leq d(x_n,x) + d(x,x_m) \leq \varepsilon$

$\therefore \{x_n\}$ is Cauchy

6. $(B(0,1), |\cdot|)$ $\quad$ $(\mathbb{Q}, |\cdot|)$ $\quad$ $(X = \{\frac{1}{n} \mid n \in \mathbb{Z}^+\}, |\cdot|)$

8. Let $\{x_n\}$ be Cauchy & $\{x_{n_k}\} \to x$

Let $\varepsilon > 0$ $\quad \exists\, N_1 \ni n,m \geq N_1 \Rightarrow d(x_n, x_m) < \frac{\varepsilon}{2}$

$\exists\, N_2 \ni n_k \geq N_2 \Rightarrow d(x_{n_k}, x) < \frac{\varepsilon}{2}$

let $N = \max(N_1, N_2)$ choose $n > N$ and let $m = n_k > N$

$$d(x_n, x) \leq d(x_n, x_m) + d(x_{n_k}, x) < \varepsilon$$

$\therefore \{x_n\} \to x$

Pg 28

1\. a) $\Rightarrow$ b)

Let $\varepsilon > 0$ $\exists\ \delta \ni x \in B_d(a,\delta) \Rightarrow f(x) \in B_\rho(f(a),\varepsilon)$

$\Rightarrow f^{-1}(B(f(a),\varepsilon)$ contains a ball centered at a

b) $\Rightarrow$ c)

Let $\varepsilon > 0$ $\exists\ \delta \ni f^{-1}(B_\rho(f(a),\varepsilon))$ contains $B_d(a,\delta)$

Since $\lim_{n\to\infty} x_n = a$ $\exists\ N \ni x_n \in B_d(a,\delta)$ if $n > N$

$\Rightarrow f(x_n) \in B(f(a),\varepsilon)$ $n \geq N$ $\Rightarrow \lim_{n\to\infty} f(x_n) = f(a)$

c) $\Rightarrow$ a) BWOC Suppose f not cont at a $\exists\ \varepsilon \ni \forall n\ \exists x_n$

$d(x_n,a) < \frac{1}{n}$ but $d(f(x_n), f(a)) \geq \varepsilon$

$\{x_n\} \to a$ $f(x_n) \not\to f(a)$ $\times$

3a f, g bdd unif cont $|f(x)| \leq M$, $|g(x)| \leq N$ $\forall x \in A$

Given ε $\exists\ \delta_f$, $\exists\ \delta_g$ $\underset{d(x,y) < \delta_f}{|f(x) - f(y)| < \varepsilon}$, $\underset{d(x,y) < \delta_g}{|g(x) - g(y)| < \varepsilon}$

$$|f(x)g(x) - f(y)g(y)| \leq |f(x)g(x) - f(y)g(x)|$$
$$+ f(y)g(x) - f(y)g(y)|$$
$$\leq N|f(x) - f(y)| + M|g(x) - g(y)|$$

$\frac{\varepsilon}{2N}\ \exists\ \delta_f$, $\frac{\varepsilon}{2M}\ \exists\ \delta_g$ let $\delta = \min(\delta_f, \delta_f)$

$d(x,y) < \delta \Rightarrow |f(x)g(x) - f(y)g(y)| < \varepsilon$

4a Yes $f: X \xrightarrow[\text{unif. cont}]{} Y$, $g: Y \xrightarrow[\text{unif cont.}]{} Z$

g unif cont $\varepsilon > 0$ $\exists \delta_g \ni d_Z(g(y_1), g(y_2)) < \varepsilon$ if $d_Y(y_1, y_2) < \delta_g$

f unif cont $\delta_g > 0$ $\exists \delta_f \ni d_Y(f(x_1), f(x_2)) < \delta_g$ if $d_X(x_1, x_2) < \delta_f$

$\therefore$ $d_X(x_1, x_2) < \delta_f \Rightarrow d_Y(f(x_1), f(x_2)) < \delta_g \Rightarrow d_Z(g(f(x_1)), g(f(x_2))) < \varepsilon$

5. $f: X \xrightarrow[\text{unif cont.}]{} \Omega$ $\{x_n\} \subset X$ Cauchy

Let $\varepsilon > 0$ $\exists$ $\delta \ni d(x, \tilde{x}) < \delta \Rightarrow \rho(f(x), f(\tilde{x})) < \varepsilon$

Given δ $\exists N \ni d(x_n, x_m) < \delta$ whenever $n, m \geq N$

$\therefore$ $n, m > N \Rightarrow \rho(f(x_n), f(x_m)) < \varepsilon$

6. Define $g: X \to \Omega$ $\begin{cases} x \in D & g(x) = f(x) \\ x \notin D & g(x) = \lim f(x_n) \text{ for } \{x_n\} \subset D,\ x_n \to x \end{cases}$

Well defined Sp $\{x_n\} \to x$ by #5 $\{f(x_n)\} \longrightarrow \alpha$
$\{y_n\} \to x$ Ω complete $\{f(y_n)\} \longrightarrow \beta$

BWOC $\alpha \neq \beta$ let $\varepsilon = \rho(\alpha, \beta) > 0$

$$\rho(\alpha, \beta) \leq \rho(\alpha, f(x_n)) + \rho(f(x_n), f(y_m)) + \rho(f(y_m), \beta))$$

$\exists N \ni n > N \Rightarrow \rho(\alpha, f(x_n)) < \frac{\varepsilon}{3}$

$\exists M \ni m \geq M \Rightarrow \rho(\beta, f(y_m)) < \frac{\varepsilon}{3}$

f unif cont $\exists\ \delta \ni\ \rho(f(a), f(b)) < \frac{\varepsilon}{3}$ when
on D

$d(a,b) < \delta$
$a, b \in D$

$x_n \to x \quad \exists\ N_1 \quad d(x_n, x) < \frac{\delta}{2} \quad n > N_1$
$y_m \to y \quad \exists\ M_1 \quad d(y_m, x) < \frac{\delta}{2} \quad m > M_1$

$P_1 = \max(N_1, M_1) \quad k > P \quad d(x_k, y_k) < \delta$
$P = \max(N, M) \quad Q = \max(P, P_1)$
$k > Q$

$\rho(\alpha, \beta) < \varepsilon \quad \cancel{*} \quad g$ well defined

Let $x, y \in X$ $\quad \exists \ \{x_n\} \subset D \quad x_n \to x$

$\{y_n\} \subset D \quad y_n \to y$

$\Rightarrow \quad \{f(x_n)\} \to g(x)$

$\{f(y_n)\} \to g(y)$

For $\varepsilon > 0$ let δ be unif cont constant for f related to $\frac{\varepsilon}{3}$

Choose $x, y \ni \ d(x,y) < \frac{\delta}{3}$

Choose $N_1 \ni \ n > N_1 \Rightarrow d(x_n, x) < \frac{\delta}{3}$

$N_2 \ni \ n > N_2 \Rightarrow d(y_n, y) < \frac{\delta}{3}$

$N = \max(N_1, N_2) \quad n > N \Rightarrow d(x_n, y_n) < \delta$

Choose $M_1 \ni \ n > M_1 \Rightarrow \rho(f(x_n), g(x)) < \frac{\varepsilon}{3}$

$M_2 \ni \ n > M_2 \Rightarrow \rho(f(y_n), g(y)) < \frac{\varepsilon}{3}$

$M = \max(M_1, M_2)$

$P = \max(N, M)$

for $d(x,y) < \frac{\delta}{3}$
$n > P$

$\rho(g(x), g(y)) \le$

$\rho(g(x), f(x_n)) +$

$\rho(f(x_n), f(y_n))$

$\rho(f(y_n), g(y)) \ < \ \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$

10 Sp $f \equiv g$ on D

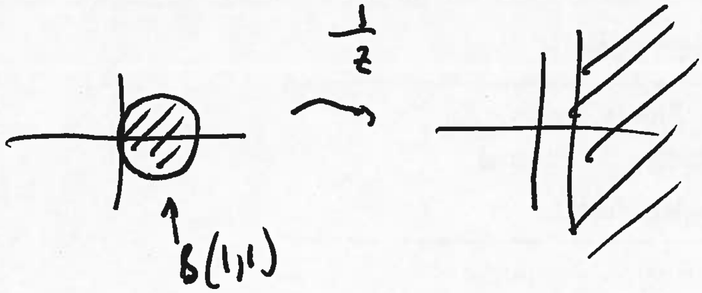
let $x \in \mathcal{X}$ $\exists \{x_n\} \subset D \ni x_n \to x$

$$\rho(f(x), g(x)) \le \rho(f(x), f(x_n)) + \rho(f(x_n), g(x_n)) + \rho(g(x_n), g(x)) < \frac{\varepsilon}{3} + 0 + \frac{\varepsilon}{3} < \varepsilon$$

$\exists N_1$ $n > N_1 \Rightarrow$ $\rho(f(x), f(x_n)) < \varepsilon/3$

$\exists N_2$, $n > N_2 \Rightarrow$ $\rho(g(x_n), g(x)) < \varepsilon/3$

P1a



P2

$N = [0,1] \setminus M$

Claim N open

Pick $x \in N$ $\quad x = 0.x_1 x_2 x_3 \ldots$

then let x_j first digit $\ni$ x_j not odd

$$x = 0.x_1 x_2 \cdots x_{j-1}\, e\, x_{j+1}' x_{j+2}' \ldots$$

WoLog not all $x_{j+k} = 0 \quad k = 1,2,3,\ldots$

$$x_- = 0.x_1 x_2 \cdots x_{j-1}\, e\, 0000$$

$$x_+ = 0.x_1 x_2 \cdots x_j (e+1) 0000$$

$$x \in (x_-, x_+) \subset N$$

P3

Since $z_n \to y$ for $\varepsilon > 0$ $\exists N$ $\quad n > N \Rightarrow |z_n - y| < \frac{\varepsilon}{2}$

Let $M = \max\{|z_1 - y|, |z_2 - y|, \ldots, |z_N - y|\}$

$n > \frac{NM2}{\varepsilon}$

$$|z_n' - y| \le \sum_{k=1}^{N} \frac{|z_k - y|}{n} + \sum_{k=N+1}^{n} \frac{|z_k - y|}{n}$$

$$\le \frac{(N)M}{n} + \frac{(n-N)}{n}\frac{\varepsilon}{2} \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$