Pg 2

1a $z = x+iy \quad \frac{1}{z} = \frac{1}{x+iy}\,\frac{x-iy}{x-iy} = \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2}$

f. $\frac{-1-i\sqrt{3}}{2} = \text{cis}\,\frac{4}{3}\pi \quad \left(\frac{-1-i\sqrt{3}}{2}\right)^6 = \text{cis}\,6\left(\frac{4}{3}\pi\right) = \text{cis}\,8\pi = 1+0i$

2e $\left|\frac{i}{i+3}\right| = \frac{|i|}{|i+3|} = \frac{1}{\sqrt{10}} \quad \overline{\frac{i}{i+3}} = \frac{\bar{i}}{\overline{i+3}} = \frac{-i}{-i+3} = \frac{-i(i+3)}{10}$

$= \frac{1-3i}{10}$

f $|(1+i)^6| = |1+i|^6 = (\sqrt{2})^6 \quad \overline{(1+i)^6} = \left(\overline{1+i}\right)^6 = (1-i)^6$

$= 8 \quad = (\sqrt{2})^6 i = 8i$

3. Let $z \in \mathbb{R} \Rightarrow z = x+i0 \Rightarrow \bar{z} = x-i0 \Rightarrow z = \bar{z}$

Let $z = \bar{z}$ where $z = x+iy \Rightarrow x+iy = x-iy \Rightarrow y = -y \Rightarrow y = 0$

$\Rightarrow z = x+i0 \Rightarrow z \in \mathbb{R}$

4c $|z+w|^2 + |z-w|^2 = (z+w)(\overline{z+w}) + (z-w)(\overline{z-w})$

$= (z+w)(\bar{z}+\bar{w}) + (z-w)(\bar{z}-\bar{w})$

$= z\bar{z} + \cancel{z\bar{w}} + \cancel{\bar{z}w} + w\bar{w} + \bar{z}z - \cancel{z\bar{w}} - \cancel{\bar{z}w} + w\bar{w}$

$= 2|z|^2 + 2|w|^2 = 2(|z|^2+|w|^2)$

6. Let $R(z) = \frac{p(z)}{q(z)}$ where $p(z), q(z)$ are polynomials.

Lemma Let $p(z)$ be a polynomial. If coeff. p are real, then $\overline{p(z)} = p(\bar{z})$

Pf. $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots a_1 z + a_0 \Rightarrow$

$\overline{p(z)} = \overline{a_n z^n + a_{n-1} z^{n-1} + \cdots a_1 z + a_0} \underset{3,5}{=} \bar{a}_n (\bar{z})^n + \bar{a}_{n-1} (\bar{z})^{n-1} + \cdots + \bar{a}_1 \bar{z} + \bar{a}_0$

$\underset{hyp}{=} a_n (\bar{z})^n + a_{n-1} (\bar{z})^{n-1} + \cdots a_1 \bar{z} + a_0 = p(\bar{z})$

Pf (of 6) $\overline{R(z)} = \overline{\frac{p(z)}{q(z)}} = \frac{\overline{p(z)}}{\overline{q(z)}} = \frac{p(\bar{z})}{q(\bar{z})} = R(\bar{z})$

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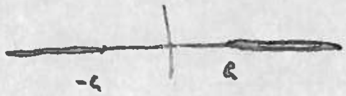
3. Describe loci $|z-a| - |z+a| = 2c$ for $a \in \mathbb{R}$, $c>0$

i) $|a| < c$ $\quad \big||z-a| - |z+a|\big| \le |(z-a)-(z+a)| = |-2a| < 2c$

$\Rightarrow$ loci $\emptyset$

ii) $|a| = c$ $\quad$ I. $a = c$

$\big||z-a| - |z+a|\big| \le |(z-a)-(z+a)| = |-2a| = 2a = 2c$

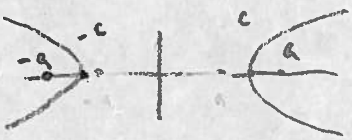


equal implies $(z-a) = k(z+a)$ $\quad k>0 \Rightarrow z \le -a$

II $a = -c$ $\quad$ MM $\quad z \ge a$

iii) $|a| > c$ $\quad$ I $a > c$

$|z-a| - |z+a| = 2c$ "left branch" of hyperbola with foci at $a, -a$ and fixed diff $2c$



II $-a > c$ $\quad$ MM $\quad$ "right branch"

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1. $\operatorname{cis}\left(\frac{2\pi}{6}\right), \operatorname{cis}\left(\frac{4\pi}{6}\right), \operatorname{cis}\left(\frac{6\pi}{6}\right), \operatorname{cis}\left(\frac{8\pi}{6}\right), \operatorname{cis}\left(\frac{10\pi}{6}\right), \operatorname{cis}\left(\frac{12\pi}{6}\right)$

2. a) $1 \operatorname{cis}\left(\frac{\pi}{4}\right), 1 \operatorname{cis}\left(\frac{\pi}{4} + \frac{2\pi}{2}\right)$

b) $1 \operatorname{cis}\left(\frac{\pi}{6}\right), 1 \operatorname{cis}\left(\frac{\pi}{6} + \frac{2\pi}{3}\right), 1 \operatorname{cis}\left(\frac{\pi}{6} + \frac{4\pi}{3}\right)$

c) $\sqrt{3} + 3i = 2\sqrt{3} \operatorname{cis}\left(\frac{\pi}{3}\right)$

$\sqrt{2\sqrt{3}} \operatorname{cis}\left(\frac{\pi}{6}\right), \sqrt{2\sqrt{3}} \operatorname{cis}\left(\frac{\pi}{6} + \frac{2\pi}{2}\right)$

5. $1 + z + z^2 + \cdots z^{n-1} = \dfrac{1 - z^n}{1 - z}$ $\quad$ if $z = \operatorname{cis} \frac{2\pi}{n}$, then $z^n = 1 \Rightarrow \blacksquare$

7. By contra. Suppose $\operatorname{Im} z > 0$, i.e., $z = |z| e^{i\alpha}$ where $0 < \alpha \le \pi$

$\exists\, n > 0 \ni \pi \ge n\alpha > \frac{\pi}{2} \Rightarrow \operatorname{Re} z^n = |z|^n e^{in\alpha} < 0$ $*$

Pg 10

2. $z = 0 \Rightarrow Z = (0, 0, -1)$

$z = 1+i \Rightarrow Z = \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$

$z = 3+2i \Rightarrow Z = \left(\frac{6}{14}, \frac{4}{14}, \frac{12}{14}\right)$

4. $\lambda = \{ z \in \mathbb{C} \mid Z \in \Lambda \}$ if $z \in \lambda$, then $\frac{2x}{|z|^2+1}\beta_1 + \frac{2y}{|z|^2+1}\beta_2 + \frac{|z|^2-1}{|z|^2+1}\beta_3 = \ell$

$z = x+iy$

$\Rightarrow 2x\beta_1 + 2y\beta_2 + (|z|^2-1)\beta_3 = (|z|^2+1)\ell$

Suppose $N \in \Lambda \Rightarrow \beta_3 = \ell \Rightarrow 2x\beta_1 + 2y\beta_2 = 2\beta_3$ eq. line

Suppose $N \notin \Lambda \Rightarrow \beta_3 \neq \ell \Rightarrow (\beta_3 - \ell)x^2 + (\beta_3 - \ell)y^2 + 2x\beta_1 + 2y\beta_2 = \beta_3 + \ell$

Pg 13

2.
a) $B(0,1)$ open
b) $\mathbb{R}$ closed
c) $\{ z \mid z^n = 1 \}$ neither
d) $[0,1)$ neither
e) $[0,1]$ closed

7. $(\mathbb{C}_\infty, d_\infty)$ metric

1. $z, z' \in \mathbb{C}$ $\quad d(z,z') \geq 0$ by def.
   $z \in \mathbb{C}, \infty$ $\quad d(z,\infty) \geq 0$

2. $d(z,z') = 0 \Rightarrow |z - z'| = 0 \Rightarrow z = z'$
   $d(z,\infty) = 0 \Rightarrow z = \infty \Rightarrow \infty = \infty$

3. $d(z,z') = d(z',z)$

4. d satisfies T.I. on $\mathbb{R}^3 \Rightarrow$ satisfies T.I. on $S \subset \mathbb{R}^3$

10 c $\quad$ Sp $x \in \text{int}(A) \Rightarrow x \in G \ni G$ open subset of A

$\Rightarrow \exists\, \varepsilon > 0 \quad B(x, \varepsilon) \subset G \subset A$

Sp $\exists\, \varepsilon \ni B(x, \varepsilon) \subset A \Rightarrow G = B(x, \varepsilon) \in \mathcal{G}$ = open subset of A

$\Rightarrow x \in \text{int } A$

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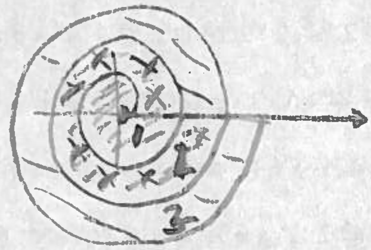
3. a) $\overline{B(0,1)} \cup B(2,1)$ $\quad$ connected

b) $[0,1] \cup \{1 + \frac{1}{n},\ n \geq 1\}$ $\quad$ not connected

$[0,1],\ \{2\},\ \{\frac{3}{2}\},\ \{\frac{4}{3}\},\ \{\frac{5}{4}\}, \dots$

c) $\mathbb{C} \setminus \{A \cup B\}$ $\quad$ not connected



P. 1. $\quad$ Sp $\exists$ ordering, i.e., $\exists\, P \subset \mathbb{C} \ni$

1. $x, y \in P \Rightarrow xy \in P$
2. $x, y \in P \Rightarrow x+y \in P$
3. $x \in P$ or $x = 0$ or $-x \in P$

a) if $i \in P$, then $i^2 \in P \Rightarrow -1 \in P$

or

if $-i \in P$, then $(-i)^2 \in P \Rightarrow -1 \in P$

b) if $-1 \in P$, then $(-1)^2 \in P \Rightarrow 1 \in P$

contradiction $-1 \in P$ or $1 \in P$