Part A. Do any five (5) of the following problems:

1. (12) Let \( z = -4 - 4i \) and \( w = 2 + 2\sqrt{3}i \). Write in rectangular form, \( a + bi \), and polar form, \( r \text{ cis } \theta \), each of the following:
   
   a. \( \frac{w^3}{z^4} \)  
   b. \((z - w)(w - \overline{z})\)

2. (12) Prove the following proposition: Let \((X, d)\) be a metric space and let \( A \) be a subset of \( X \). Then, \( \partial A = \overline{A} \setminus \text{int } A \).

3. (12) Prove the following proposition: Let \((X, d)\) be a complete metric space and let \( Y \subset X \). If \( Y \) is closed, then \( Y \) is complete.

4. (12) Prove the following proposition: Let \( z, w \in \mathbb{C}, w \neq 0 \). Then, \( \frac{\overline{z}}{w} = \frac{\overline{w}}{z} \)

5. (12) Give examples of sequences of subsets \( \{F_n\} \) of \( \mathbb{C} \) such that
   
   a. each \( F_n \) is closed and for each \( n \geq 1 \), \( F_n \supset F_{n+1} \), but \( \bigcap_{n=1}^{\infty} F_n \) is not a singleton
   
   b. each \( F_n \) is closed and \( \lim_{n \to \infty} \text{diam}(F_n) = 0 \) but \( \bigcap_{n=1}^{\infty} F_n \) is not a singleton
   
   c. for each \( n \geq 1 \), \( F_n \supset F_{n+1} \), and \( \lim_{n \to \infty} \text{diam}(F_n) = 0 \), but \( \bigcap_{n=1}^{\infty} F_n \) is not a singleton

6. (12) Prove the following proposition: Let \((X, d)\) be a metric space. If \( C \) is a component of \( X \), then \( C \) is closed.
Part B. Do each of the following problems:

7. (15) Provide a counterexample to each of the following assertions:

a. In a metric space \((X, d)\), for any \(A \subseteq X\) we have \(diam(int(A)) = diam(A)\)

b. In a metric space \((X, d)\), for any \(A \subseteq X\) we have \(\text{int } \overline{A} \subseteq \text{int } A\)

c. Let \((X, d)\) be a metric space and let \(f : X \rightarrow \mathbb{R}\) be continuous on \(X\). If \(A\) is a closed subset of \(X\), then \(f(A)\) is a closed subset of \(\mathbb{R}\).

d. Let \((X, d)\) be a compact metric space and let \(f : X \rightarrow \mathbb{R}\) be continuous on \(X\). Then, \(f\) is Lipschitz.

e. Let \((X, d)\) be a metric space and let \(f : X \rightarrow \mathbb{R}\) be bounded and continuous on \(X\). Then, \(f\) is uniformly continuous.

8. (30) Classification Problem. Correctly identify whether the following subsets of \(\mathbb{C}\) are: (a) open; (b) closed; (c) connected; (d) polygonally path connected; (e) compact; (f) complete; (g) bounded; (h) region. You do not need to provide a rationale for your classification. Fill out the classification information on the attached page, i.e., if the set possesses the property **mark the cell** in the table with \(Y\) (= yes) if the set does not possess the property **do not mark** the cell in the table

A. \(\{B(0,1) \setminus B(\frac{1}{4}, \frac{1}{2})\} \cap \{z : \Re z > 0\}\)

B. \(\{B(0,1) \setminus B(\frac{1}{2}, \frac{1}{2})\}\)

C. \(T \setminus B(0,1)\) where \(T\) is the equilateral triangle (interior plus the sides) centered at 0 with vertices \(\left\{2, 2\text{cis} \frac{2\pi}{3}, 2\text{cis} \frac{4\pi}{3}\right\}\). Note: the circle \(C(0,1)\) is inscribed inside of \(T\) and is tangent to the side of \(T\) at the points \(\{-1, \text{cis} \frac{\pi}{3}, \text{cis} -\frac{\pi}{3}\}\).

D. \(\{z = x + iy : xy < 2, xy > 0\} \setminus B(0,4)\)

E. \(\bigcup_{n=1}^{\infty} l_n\), where each \(l_n = [0, b_n]\) where \(b_n = \text{cis} \frac{\pi}{2n}\)
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