Math 5320 | Take-Home Exam I | 27 September 2013

Due: October 2

Work independently. Answer the problems on separate paper. You do not need to rewrite the problem statements on your answer sheets. Work carefully. **Show all relevant supporting steps!**

1. (10 pts) Prove that if $\operatorname{Re} z > 0$ then $\left|\frac{z-2}{z+2}\right| < 1$.

2. (10 pts) For each of the following cases, give an example of a metric space (X, d), where $X \subset \mathbb{C}$ and d is the inherited metric such that

   a. there exists a sequence $\{z_n\} \subset X$ such that no subsequence of $\{z_n\}$ converges.

   b. there exists a sequence $\{z_n\} \subset X$ such that $\{z_n\}$ is Cauchy, but $\{z_n\}$ does not converge.

   c. there exists a sequence $\{z_n\} \subset X$ such that there exists two subsequences of $\{z_n\}$, say $\{z_{n_j}\}$ and $\{z_{n_k}\}$, such that $\{z_{n_j}\} \to z'$ and $\{z_{n_k}\} \to z''$ and $z' \neq z''$.

   d. every point of X is a limit point of X.

   e. X has exactly three limit points.

3. (10 pts) Sketch each of the following subsets T of $\mathbb{C}$:

   a. $T = \{w : w = \frac{1}{z}, z \in S\}$ where $S = \{z : \operatorname{Re} z > 0, \operatorname{Im} z < 0, |z| < 4\}$

   b. $T = \{z : |z| - \operatorname{Im}(z) < 1\}$

4. (10 pts) Let $G \subset \mathbb{C}$, G open, and let $f : G \to \mathbb{C}$, $f(z) = u(x, y) + i\,v(x, y), z = x + iy$. Prove that if f is continuous on G, then u and v are continuous on G.

5. (10 pts) For each of the following cases, give an example such that the described properties holds or if such an example does not exist then state so:

a. Let A, B be connected subsets of $\mathbb{C}$ such that $A \cap B \neq \varnothing$ and $A \cap B$ is not connected

b. Let A, B be connected subsets of $\mathbb{C}$ such that $A \cap B \neq \varnothing$ and $A \cap B$ is connected

c. Let A, B be subsets of $\mathbb{C}$ such that $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$

d. Let A, B be subsets of $\mathbb{C}$ such that $\overline{A \cup B} \neq \overline{A} \cup \overline{B}$

e. Let $\{x_n\}$ be a sequence in $\mathbb{R}$ such that $\liminf x_n = \limsup x_n$ but $\lim x_n$ does not exist.