Answer the problems on separate paper. You do not need to rewrite the problem statements on your answer sheets. Work carefully. Do your own work. **Show all relevant supporting steps!**

1. Let \( z = \sqrt{3} - i \) and \( w = -2 + 2i \). Find \( \frac{w^3}{z^9} \) and express the value in rectangular form.

2. Let \((X,d)\) be a metric space and let \( A \subseteq X \). Prove: \( A \) is closed if and only if \( A = \overline{A} \).

3. Classify the following sets as to whether they are: a) open, b) closed, c) connected, d) polygonally path-connected, e) compact, f) complete, g) bounded, h) region.

You do not need to provide a rationale for your classification. Use the table on the last page to record your classifications.

1. \( A = \{(x,y) : 0 < y < \frac{1}{x} |\sin\frac{1}{x}|, 0 < x < \pi\} \)
2. \( B = B(1,1) \setminus B\left(\frac{1}{2}, \frac{1}{2}\right) \)
3. \( C = S \setminus B(0,\pi) \) where \( S = \{z : |\text{Re}z| < 1\} \)
4. \( D = B(2,2) \setminus B(2 - 2i, \sqrt{8}) \)

4. Find the radius of convergence of the power series

1. \( \sum_{n=0}^{\infty} \frac{2^n n! n!}{(2n)!}(3z + 1)^n \)
2. \( \sum_{n=0}^{\infty} a_n (z + i)^n \) where the power series represents \( f(z) = \tan z \)

5. Let \( G = \{z = re^{i\theta} : 0 < r < 1, \ 0 < \theta < \frac{\pi}{2}\} \) and let \( S = \{z : |\text{Re}z| < 1\} \). Find a one-to-one conformal map from \( G \) to \( S \).

6. The line \( L = \{z : \text{Im}z = 0\} \) and the circle \( C_1 = \{z : |z - 1| = 1\} \) divide \( D = \{z : |z| < 1\} \) into 4 subregions \( D_1, D_2, D_3, \) and \( D_4 \). See figure to the right. Let \( w = \frac{z}{z-2} \). Find the images \( E_j \) of each subregion \( D_j \) under \( w \), i.e., find \( E_j = w(D_j), \) \( j = 1, 2, 3, 4 \).

7. Let \( G \) be a region in \( \mathbb{C} \) and let \( f \in \mathcal{A}(G), f = u + iv \). Prove that if \( u^2 - v^2 = 1 \) on \( G \), then \( f \) is constant on \( G \).
8. Evaluate each of the following integrals:

1. \[ \int_\gamma (\text{Re } z + (\text{Im } z)^2) \, dz \] where \( \gamma \) is a parametrization of the straight line segment from 1 to \( i \).

2. \[ \int_\gamma \frac{\cos z + 1}{z^3} \, dz \] where \( \gamma \) is a positively (counter-clockwise) oriented parametrization of the unit circle \( C = \{ z : |z| = 1 \} \)

3. \[ \int_\gamma \frac{z + 1}{z^2 - 2z} \, dz \] where \( \gamma \) is a parametrization of the figure eight curve oriented as given in the figure to the right.

4. \[ \int_\gamma \frac{dz}{\sqrt{z}} \] where \( \gamma \) is a parametrization of the arc \( \Gamma \) of the unit circle \( C = \{ z : |z| = 1 \} \) from \( i \) to \(-i\) passing through \(-1\) (see figure to the right) and where \( \sqrt{z} \) is the branch of square root \( z \) chosen so that the branch cut is taken as the positive real axis, \([0, \infty)\), such that \( \sqrt{-1} = i \)

5. \[ \int_\gamma \frac{1 + z}{e^z - e^{-z}} \, dz \] where \( \gamma \) is a positively (counter-clockwise) oriented parametrization of the circle \( C_2 = \{ z : |z - 2| = 1 \} \)

9. Prove that if \( f \) is an entire function and if there exist positive constants \( A \) and \( B \) such that \[ |f(z)| \leq A |z| + B \] for all \( z \in \mathbb{C} \), then \( f \) is a linear function.

10. Let \( D = \{ z : |z| < 1 \} \) and let \( f \in \mathcal{A}(D) \) such that \( f(D) \subset D \). Prove for \( z \in D \) that \[ |f'(z)| \leq \frac{1}{1 - |z|} \]. Hint. Suppose \( z \in D \). Then, there exists a \( \rho > 0 \) such that \( B(z, \rho) \subset D \).

11. Let \( D = \{ z : |z| < 1 \} \) and let \( f \in \mathcal{A}(D) \) such that \( f(D) \subset D \). Suppose that for each integer \( n > 1 \) that \( f \) satisfies \( f\left(\frac{i}{n}\right) = -\frac{i}{n^3} \). Find the value of \( f\left(\frac{1}{2} + \frac{1}{2}i\right) \).

12. State and prove Liouville’s Theorem.
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