Math 5320 Exam I September 29, 2008

Make-up

Answer the problems on separate paper. You do <u>not</u> need to rewrite the problem statements on your answer sheets. Work carefully. Do your own work. **<u>Show all relevant supporting steps!</u>** You may keep the exam sheet for your records.

Part A. Do any five (5) of the following problems:

1. (12) Let $z = -2\sqrt{3} - 2i$ *and* $w = -3 + 3i$. Write in rectangular form, $a + b\,i$, each of the following:

   a. $\dfrac{w^4}{z^3}$ b. $(z - \overline{w})(\overline{z} - w)$

2. (12) Prove the following proposition: Let $z, w, \zeta \in \mathbb{C}$.
   Then: $|\, z + w + \zeta \,| \;\le\; |\, z \,| + |\, w \,| + |\, \zeta \,|$.

3. (12) Prove the following proposition: Let A be a subset of a metric space (X , d).
   Then: A is closed if and only if $A = \overline{A}$.

4. (12) Prove the following proposition: Let (X , d) be a complete metric space and let $Y \subset X$.
   Then: If Y is closed in X , then (Y , d) is a complete metric space.

5. (12) Prove the following proposition: Let A be a subset of a metric space (X , d).
   Then: $\operatorname{diam}(A) = \operatorname{diam}(\overline{A})$

6. (12) Prove the following proposition: Let $f\,, g : \mathbb{R} \to \mathbb{R}$ be uniformly continuous.
   Then: $f + g$ is uniformly continuous.

Part B. Do each of the following problems:

7. (10) Provide a counterexample to each of the following assertions:

a. In the metric space $(\mathbb{C}, |\cdot|)$, for any $A \subset \mathbb{C}$ we have:
If A is polygonally path connected, then diam(int(A)) = diam(A)

b. In a metric space (X , d), for any $x \in X$ and $A \subset X$ we have:
If $\operatorname{int} A \neq \varnothing$, then $d(x, \overline{\operatorname{int} A}) = d(x, A)$.

c. Let $\mathbb{R}^* = \{x \in \mathbb{R} : x \geq 0\}$. Let $f : \mathbb{R}^* \to \mathbb{R}^*$ such that (1) $f(0) = 0$,
(2) f is strictly increasing, (3) f is twice continuously differentiable on $\mathbb{R}^*$.
Let $X = \mathbb{R}$. For $x, y \in X$ define $d(x, y) = f(|x - y|)$. Then, (X, d) is a metric space.

d. Let (X , d) be a metric space and let $f : X \to \mathbb{R}$ be bounded and continuous on X. Then, f is uniformly continuous.

e. Let (X , d) be a metric space and let $f : X \to \mathbb{R}$ be continuous on X. If A is a closed subset of X, then $f(A)$ is a closed subset of $\mathbb{R}$.

8. (30) Classification Problem. Correctly identify whether the following subsets of $\mathbb{C}$ are:
(a) open; (b) closed; (c) connected; (d) polygonally path connected; (e) compact;
(f) complete; (g) bounded; (h) region. You do not need to provide a rationale for your classification. Fill out the classification information on the attached page.

A. $\{z = x + iy : x > 0, |y| > \frac{1}{|x|}\}$

B. $\{B(3,2) \cup B(-3,2)\} \setminus \overline{B(0,2)}$

C. $T \setminus B(0,1)$ where T is the equilateral triangle (interior plus the sides) centered at 0 with vertices $\left\{2, 2\operatorname{cis}\frac{2\pi}{3}, 2\operatorname{cis}\frac{4\pi}{3}\right\}$. Note: the circle C(0,1) is inscribed inside of T and is tangent to the side of T at the points $\{-1, \operatorname{cis}\frac{\pi}{3}, \operatorname{cis}\frac{-\pi}{3}\}$.

D. $B(0,4) \cap S$ where $S = \{z : |\operatorname{Re} z| \leq 2\}$

E. $\bigcup_{n=1}^{\infty} B_n$, where each $B_n = \overline{B(c_n, \frac{1}{n})}$ and the centers c_n satisfy

$c_1 = 1, c_2 = c_1 + 1 + \frac{1}{2}, c_3 = c_2 + \frac{1}{2} + \frac{1}{3}, \cdots, c_{n+1} = c_n + \frac{1}{n} + \frac{1}{n+1}$ for $n \geq 1$

Classification Table for Problem 8

	open	closed	connected	polygonally path connected	compact	complete	bounded	region
A								
B								
C								
D								
E								