MATH 5320 Exam I October 4, 2004

In-Class Make-Up

Due Friday, Oct 8

Answer the problems on separate paper. You do <u>not</u> need to rewrite the problem statements on your answer sheets. Work carefully. Do your own work. **<u>Show all relevant supporting steps!</u>**

1. (10) Let $z = -2\sqrt{3}\, i - 2$ and $w = 4 - 4i$. Write in rectangular form, $a + b\,i$, and polar form, $r \operatorname{cis} q$, each of the following:

   a. $\dfrac{z^2}{w^3}$ b. $\left(\dfrac{z + \overline{w}}{\overline{z} + w}\right)^3$

2. (10) Prove Proposition 1.13 d.

3. (22) Give examples of sets in $\mathbb{C}$ with the ususal topology:

   a. i) A set A such that $\operatorname{int}(\overline{A} \setminus \operatorname{int} A) = \varnothing$

   ii) A set A such that $\operatorname{int}(\overline{A} \setminus \operatorname{int} A) \neq \varnothing$

   b. i) A set A such that A has only finitely many components

   ii) A set A such that A has countably infinitely many components

   iii) A set A such that A has uncountably many components

   A countable collection $\{A_n\}_{n=1}^{\infty}$ is said to be distinct if $j \neq k \Rightarrow A_j \neq A_k$ for all $j, k \in \mathbb{N}$

   c. i) A countable collection of distinct closed sets $\{G_n\}_{n=1}^{\infty}$ such that

   a) $\bigcup_{n=1}^{\infty} G_n$ is open $(\neq \mathbb{C})$

   b) $\bigcup_{n=1}^{\infty} G_n$ is closed $(\neq \mathbb{C})$

   c. $\bigcup_{n=1}^{\infty} G_n$ is neither open nor closed

ii) A countable collection of distinct open sets $\{F_n\}_{n=1}^{\infty}$ such that

a) $\bigcap_{n=1}^{\infty} F_n$ is closed $(\neq \varnothing)$

b) $\bigcap_{n=1}^{\infty} F_n$ is open $(\neq \varnothing)$

c. $\bigcap_{n=1}^{\infty} F_n$ is neither open nor closed

4. (10) Prove the following proposition: Let (X, d) be a metric space. Let $f, g : X \to \mathbb{C}$ be uniformly continuous on X. Then, $f + g$ is uniformly continuous on X.

5. (9) Give examples of sequences $\{x_n\}$, $\{y_n\}$ in $\mathbb{C}$ such that

a. $\lim x_n y_n = L$, $\lim x_n = M$, $\lim y_n$ does not exist

b. $\lim x_n y_n = L$, $\lim x_n$ does not exist, $\lim y_n$ does not exist

c. $\lim \frac{x_n}{y_n} = L$, $\lim x_n = M$, $\lim y_n = N$, $L \neq \frac{M}{N}$

6. (6) Give examples of distinct sequences $\{x_n\}$ in $\mathbb{C}$ such that

a. $\#\{x_n\}' = 5$ (i.e., the sequence has exactly five limit points}

b. the sequence $\{x_n\}$ is given by a formula $x_n = f(n)$, where $f : \mathbb{N} \to \mathbb{C}$, such that $x_1 = 1,\ x_2 = 2,\ x_3 = 4,\ x_4 = 8,\ x_5 = 16,$ but $x_6 \neq 32$

(i.e., give a closed formula for the function f defining the sequence x_n).

7. (8) For $a, b, c \in \mathbb{C}$, $a \neq 0$, show that $ab = ac \Rightarrow b = c$.

8. (6) Provide a counterexample to each of the following assertions:

a. In a metric space (X, d), if a set A is closed and bounded, then A is compact.

b. Let (X, d), (Ω, r) be metric spaces. Let $f : X \to \Omega$. If f is uniformly continuous on X, then f is Lipschitz on X.

9. (20) Classification Problem. Correctly identify whether the following subsets of $\mathbb{C}$ are: (a) open; (b) closed; (c) connected; (d) polygonally path connected; (e) compact; (f) complete; (g) bounded; (h) region. You do not need to provide a rationale for your classification. Fill out the classification information on the attach table.

A. $B(0,5)\setminus \overline{E}$ where $E = \{z = x+iy : \frac{x^2}{25} + \frac{y^2}{4} < 1\}$

B. $\overline{B(0,5)}\setminus E$ where $E = \{z = x+iy : \frac{x^2}{25} + \frac{y^2}{4} < 1\}$

C. $B(0,1)\setminus \overline{B(\frac{1}{2},\frac{1}{2})}$

D. $\{\overline{B(0,4)}\setminus B(1,1)\} \cap \{z : \mathrm{Re}\, z \geq 0\}$

E. $\bigcup_{n=1}^{\infty} l_n$, where each $l_n = [0, \frac{cis(\frac{p}{2n})}{n}]$

Classification Table for Problem 9

	open	closed	connected	polygonally path connected	compact	complete	bounded	region
A								
B								
C								
D								
E								