Notation:

$\mathbb{C}$ = the complex plane, $D = \{\, z \mid |z| < 1 \,\}$,

$RHP = \{\, z \mid Re\, z > 0 \,\}$, $UHP = \{\, z \mid Im\, z > 0 \,\}$, $Q_1 = RHP \cap UHP$,

$\mathcal{A}(G) = \{\, f : G \to \mathbb{C} \mid f \text{ is analytic on } G \}$,

$\mathcal{C}(G) = \{\, f : G \to \mathbb{C} \mid f \text{ is continuous on } G \}$

1. Let D_1 denote the region $Q_1 \setminus [0,1+i]$. Find a one-to-one, onto, conformal map f from D_1 to D.

2. Suppose that $f \in \mathcal{A}(\mathbb{C})$ satisfies $Im\, f(z) > 0$ for $z \in \mathbb{C}$. Prove that f is constant on $\mathbb{C}$.

3. Prove there does not exist a function f such that $f \in \mathcal{A}(\overline{B(1,4)})$, $f''(1) = i$ and $\max_{\overline{B(1,4)}} |f(z)| = 5$. Hint: Consider the implications of Cauchy's Estimate.

4. Let γ be rectifiable and $f \in \mathcal{C}(\gamma)$. Prove: $\left| \int_\gamma f \right| \le V(\gamma) \max_{\{\gamma\}} |f(z)|$.

5. Let G be a region and $f \in \mathcal{A}(G)$. Prove: If there exists a $B(z_0,r) \subset G$, $r > 0$, such that $Re\, f$ is constant on $B(z_0,r)$, then f is constant on all of G.

6. Evaluate $\displaystyle\int_{|z-\frac{\pi}{2}|=1} \frac{\sin^2 z}{z - z^3}\, dz$.

7. Suppose that $\displaystyle\sum_{n=0}^{\infty} a_n (z-1)^n$ is a series expansion (centered at $z = 1$) for $\dfrac{1}{\cos z}$.

   Determine whether $\displaystyle\sum_{n=0}^{\infty} |a_n|$ converges.

8. Evaluate $\displaystyle\int_\gamma \frac{dz}{2z - z^2}$ where $\gamma(t) = e^{it}$, $0 \le t \le \dfrac{3\pi}{2}$.