MATH 5320 Exam II November 17, 1997

Notation. D denotes the open unit disk centered at 0. RHP denotes the open right-half plane. Q_1 denotes the open first quadrant.

Part B. Take home.

1. Let $u(x,y) = y^2 - x^2 + 3x$ on RHP. Show that u is harmonic on RHP. Find a real-valued function $v(x,y)$ on RHP such that $f = u + iv$ is analytic on RHP.

2. Find a function f which maps D conformally, one-to-one and onto the set $D_1 = D\setminus(-1,0]$ such that $f(0) = ½$ and $f\,'(0) > 0$.

3. Let $f(re^{i\theta}) = R(r,\theta)e^{i\Phi(r,\theta)}$ be analytic on a region G. Derive the polar form of the Cauchy-Riemann equations which R and Φ satisfy.

4. Use DeMoivre's formula to show
$$\cos 5x = 16 \cos^5 x - 20 \cos^3 x + 5 \cos x$$
Derive an analogous formula for expressing $\cosh 5x$.

5. Compute $\int_{\gamma_1} f$ and $\int_{\gamma_2} f$ for $\gamma_1(t) = e^{it}$, $0 \le t \le \pi/2$ and $\gamma_2 = [1,i]$ for:

   (a) $f(z) = 1/\overline{z}$, (b) $f(z) = \{ Re\, z + Im\, z \}^2$, (c) $f(z) = |z|^2 - Im\, z^2$.

6. Let $D_1 = Q_1 \setminus \overline{D}$ and let $S = \{ z \mid -1 < \text{Im}\, z < 1 \}$. Find a one-to-one conformal onto mapping $f : D_1 \to S$ such that $f(2+2i) = 0$.

7. Definition. Let G be a region and let $f \in \mathcal{A}(G)$. Let $F : G \to \mathbb{C}$ continuous such that $e^{F(z)} = f(z)$ on G. We will say that F is a branch-of-logarithm f on G.

   Prove: (a) The totality of branches-of-logarithm f on G are the functions $F + 2\pi k i$, $k \in \mathbf{Z}$, where F is some branch-of-logarithm f on G.

   (b) If F is a branch-of-logarithm f on G, then $F \in \mathcal{A}(G)$ and $F' = f\,'/f$ on G.