Solution Set #9

Section 9.6

1. a. Let $a_n = \frac{(-1)^n}{2n+1}$, $n = 0, 1, 2, \ldots$ Let $f(x) = \sum_{n=1}^{\infty} a_n x^n$. Let $g(x) = xf(x^2)$.

Then, $g'(x) = \frac{d}{dx}\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n x^{2n} = \frac{1}{1+x^2}$. Hence,

$f(x^2) = \frac{1}{x}\int_0^x \frac{1}{1+t^2}\,dt = \frac{\arctan x}{x}$. Consequently,

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} f(x^2) = \lim_{x\to 1^-} \frac{\arctan x}{x} = \arctan 1$. Hence, the series $\sum_{n=1}^{\infty} a_n$ is Abel summable.

b. Let $a_n = (-1)^n \frac{(n+1)\,(n+2)}{2}$. Let $f(x) = \sum_{n=1}^{\infty} a_n x^n$. Then, let

$g(x) = \int_0^x f(t)\,dt = \sum_{n=0}^{\infty} (-1)^n \frac{(n+2)}{2} x^{n+1}$ and let

$h(x) = \int_0^x g(t)\,dt = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2} x^{n+2} = \frac{x^2}{2}\,\frac{1}{1+x}$. Hence,

$f(x) = \frac{d^2}{dx^2} h(x) = \frac{1}{(1+x)^3}$. Hence, $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{1}{(1+x)^3} = \frac{1}{8}$. Hence, the series $\sum_{n=1}^{\infty} a_n$ is Abel summable.

2. Let $f(x) = \sum_{n=1}^{\infty} a_n x^n$ and $g(x) = \sum_{n=1}^{\infty} b_n x^n$. By hypothesis $\lim_{x\to 1^-} f(x) = L$ and $\lim_{x\to 1^-} g(x) = M$. But, then $\lim_{x\to 1^-} (f+g)(x) = L + M$, which implies that the series $\sum_{n=1}^{\infty} a_n + b_n$ is Abel summable and $\sum_{n=1}^{\infty} a_n + b_n = L + M \quad (A)$.

3. Let $\sum_{n=1}^{\infty} a_n = L \quad (A)$. Then, $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \sum_{n=0}^{\infty} a_n x^n = L$. Let $g(x) = xf(x^2)$. Then, $\lim_{x\to 1^-} g(x) = \lim_{x\to 1^-} x f(x^2) = L$. Since the coefficients of g are $0, a_0, 0, a_1, 0, a_2, \ldots$, then we have $(0 + a_0 + 0 + a_1 + 0 + a_2 + \ldots = L$ (A).

4. Since $\sum_{n=0}^{\infty} a_n L^n$ converges, then by Theorem 9.6C we have $\sum_{n=0}^{\infty} a_n L^n x^n$ converges uniformly on [0,1]. But for any z ∈ [0,L] we can write z = xL for some x ∈ [0,1]. Hence, $\sum_{n=0}^{\infty} a_n z^n$ converges uniformly on [0,L].

Section 10.1

1. Let $f \in \mathcal{C}[a,b]$. Let ε > 0 be given. Claim if we set δ = ε/(b-a), then ρ(f,g) < δ will imply that |L(f) - L(g)| < ε . We note that

$$|L(f) - L(g)| = |\int_a^b f - \int_a^b g| \le \int_a^b |f - g| \le$$
$$\int_a^b \|f - g\| = (b - a)\|f - g\| = (b - a)\,\rho(f,g)$$

Hence, the claim holds.

4. Suppose $f \in A$. Then, since $f \in \mathcal{C}[a,b]$ we have $M_1 = \min_{x\in[a,b]} f(x)$ and $M_2 = \max_{x\in[a,b]} f(x)$. Since $f \in A$, we must have $m < M_1$ and $M_2 < n$. Let $\delta = \min\left(\frac{M_1 - m}{2}, \frac{n - M_2}{2}\right)$. Claim that the ball (in the metric of $\mathcal{C}[a,b]$) $B(f, \delta) \subset A$. Suppose that $g \in B(f, \delta)$. Then, $\| f - g \| < \delta$. That implies that $\max_{x\in[a,b]} g(x) \le \max_{x\in[a,b]} f(x) + \delta < n$. Similarly, $\min_{x\in[a,b]} g(x) \ge \min_{x\in[a,b]} f(x) - \delta > m$. Hence, $g \in A$.

Section 10.2

1. From Maple

```
> n := 1;
                    n := 1
> f := x^2;

                    f := x^2

> for k from 0 to n do p[k] := binomial(n,k)*x^k * (1-x)^(n-k)*subs(t=k/n,f) od:
> p := sum(p[i],i=0..n);

              p := (1 - x) x^2  + x^3
```

```
> n := 2;
```

$$n := 2$$

```
> f := x^2;
```

$$f := x^2$$

```
> for k from 0 to n do p[k] := binomial(n,k)*x^k * (1-x)^(n-k)*subs(t=k/n,f) od:
> p := sum(p[i],i=0..n);
```

$$p := (1 - x)^2 \; x^2 + 2\, x^3 \; (1 - x) + x^4$$

```
> n := 3;
```

$$n := 3$$

```
> f := x^2;
```

$$f := x^2$$

```
> for k from 0 to n do p[k] := binomial(n,k)*x^k * (1-x)^(n-k)*subs(t=k/n,f) od:
> p := sum(p[i],i=0..n);
```

$$p := (1 - x)^3 \; x^2 + 3\, x^3 \; (1 - x)^2 + 3\, x^4 \; (1 - x) + x^5$$