Solution Set #8

Section 9.1

1. Yes. Take $N = 11$. Then for $n \geq N$, $\left|\frac{\sin nx}{n} - 0\right| = \left|\frac{\sin nx}{n}\right| \leq \frac{1}{n} \leq \frac{1}{N} < \frac{1}{10}$.

2. a. For $0 \leq x < 1$, we know that $x^n \longrightarrow 0$ as $n \longrightarrow \infty$. Hence, for $0 \leq x < 1$, we have $f_n(x) \longrightarrow 0$. For $x = 1$, we have $x^n = 1$; so $f_n(1) = \frac{1}{2}$. Hence, $\{f_n\}$ converges pointwise on [0,1] to the function $f(x) = \begin{cases} 0 & 0 \leq x < 1 \\ \frac{1}{2} & x = 1 \end{cases}$.

   b. No. For any n, we have f_n is continuous at $x = 1$. Hence, for any n there exists a neighborhood of $x = 1$ on which $|f_n(x) - \frac{1}{2}| < \frac{1}{4}$. But, then on that neighborhood, for $x \neq 1$, we cannot have $|f_n(x) - 0| < \frac{1}{4}$.

3. a. Let $x \in [0,1]$. If $x = 0$, then $f_n(0) = 0$ for all n. Hence, $f_n(0) \longrightarrow 0$ as $n \longrightarrow \infty$. If $x > 0$, then there exists (by the Archimedian principle) an N such that for $n \geq N$ we have $1/n < x$. Hence, for $n \geq N$ we have $f_n(x) = 0$ and, hence, $f_n(0) \longrightarrow 0$ as $n \longrightarrow \infty$.

   b. No. For any n, we have that $f_n(x) = n > \frac{1}{2}$ for $0 < x < 1/n$.

   c. For each n, $\int_0^1 f_n = \int_0^{1/n} n = 1$. Hence, $\lim_{n \to \infty} \int_0^1 f_n = 1$

   d. We have from a. and c. that $\lim_{n \to \infty} \int_0^1 f_n \neq \int_0^1 \lim_{n \to \infty} f_n$

Section 9.2

1. Let $\{f_n\}$ converge uniformly on E to f and let $\{g_n\}$ converge uniformly on E to g. We claim that $\{f_n + g_n\}$ converges uniformly on E to $f + g$. Let $\varepsilon > 0$ there exists N_1 such that for $n > N_1$ we have $|f_n(x) - f(x)| < \varepsilon/2$ for all $x \in E$ and there exists N_2 such that for $n > N_2$ we have $|g_n(x) - g(x)| < \varepsilon/2$ for all $x \in E$. Let $N = \max(N_1, N_2)$. Then, for $n > N_2$ we have $|(f_n + g_n)(x) - (f + g)(x)| \leq |f_n(x) - f(x)| + |g_n(x) - g(x)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$ for all $x \in E$. Hence, $\{f_n + g_n\}$ converges uniformly on E to $f + g$.

2. On $[0, \infty)$ we have that $|g_n(x)| \leq g_n(0) = 1/n$ since g_n is a non-negative decreasing function of x. The sequence $\{1/n\}$ converges monotonically to 0. Let $\varepsilon > 0$ there exists N such that for $n > N$ we have $1/n < \varepsilon$. Hence, for $n > N$ we have $|g_n(x) - 0| = |g_n(x)| \leq g_n(0) = 1/n < \varepsilon$ for all x in $[0, \infty)$. Thus, $\{g_n\}$ converges uniformly to 0 on $[0,\infty)$.

4. a. On $[0,½]$ we have that $| f_n(x) | \le f_n(½) = \dfrac{1}{2^n + 1}$ since f_n is a non-negative increasing function of x on $[0,½]$. Since the sequence $\{ \dfrac{1}{2^n + 1} \}$ converges converges monotonically to 0, we have by the same argument as employed in 2. above that $\{ f_n \}$ converges uniformly to 0 on $[0,½]$.

b. The conclusion of problem 2b. in Section 9.1 is that for $\varepsilon = ¼$ there does **not** exist a N such that for $n \ge N$ we can have $| f_n(x) - 0 | < \varepsilon$ on $[0,1)$. Hence, we cannot have uniform convergence on $[0,1]$.

Section 9.3

1. If the sequence $\{ f_n \}$ were to be uniformly convergent on $[0,\infty)$, then since each f_n is continuous on $[0,\infty)$ we would have to have the limit function continuous on $[0,\infty)$. But, the limit function is $f(x) = \begin{cases} 1 & 0 \le x < 1 \\ \dfrac{1}{2} & x = 1 \\ 0 & x > 1 \end{cases}$ which is not continuous at x = 1. Hence, we cannot have uniform convergence on $[0,\infty)$.

Section 9.4

1. a. Apply the Weierstrass M-Test with $M_n = 1/n^2$. Then, on $[0,\infty)$ we have $\dfrac{1}{n^2 + x^2} \le \dfrac{1}{n^2} = M_n$. Since the series $\sum_{n=1}^{\infty} \dfrac{1}{n^2}$ converges, then by the Weierstrass M-Test we have $\sum_{n=1}^{\infty} \dfrac{1}{n^2 + x^2}$ converges uniformly on $[0,\infty)$.

3. Apply the Weierstrass M-Test with $M_n = | a_n |$. Then, on $[0,1]$ we have $| a_n x^n | \le | a_n | = M_n$. Since by hypothesis series $\sum_{n=1}^{\infty} | a_n |$ converges, then by the Weierstrass M-Test we have $\sum_{n=1}^{\infty} a_n x^n$ converges uniformly on $[0,1]$.

4. If the series $\sum_{n=1}^{\infty} a_n$ converges, then the power series $f(x) = \sum_{n=1}^{\infty} a_n x^n$ converges for $x = x_0 = 1$. By Theorem 9.4F the power series converges uniformly on $[-x_1 , x_1]$ for all x_1 such that $0 < x_1 < x_0 = 1$. Hence, on each such interval $[-x_1 , x_1]$ with $0 < x_1 < x_0 = 1$ the power series f is a continuous function. But every x in $(-1,1)$ belongs to an interval of the form $[-x_1 , x_1]$ with $0 < x_1 < x_0 = 1$, namely the interval $[-x,x]$.

Section 9.5

1. By the above problem 9.4.3, the power series $\sum_{n=1}^{\infty} a_n x^n$ converges uniformly on [0,1].

   Hence, by Theorem 9.5A $\int_0^1 \sum_{n=0}^{\infty} a_n x^n \, dx = \sum_{n=0}^{\infty} \int_0^1 a_n x^n \, dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1}$.

2. Since the power series $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$ has radius of convergence of infinity, i.e., since the power series converges on all intervals (-*S* , *S*) for all *S* > 0, then by Theorem 9.5C the sin *x* is differentiable on all intervals (-*S* , *S*) for all *S* > 0, and

   $\cos x = (\sin x)' = 1 - 3\frac{x^2}{3!} + 5\frac{x^4}{5!} - \cdots = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$ on $(-\infty, \infty)$.