Solution Set #7

Section 8.5

1. Let $f(x) = x^3 + 2x + 1$ and center a = 2. Then,

n	$f^{(n)}(x)$	$f^{(n)}(2)$	$f^{(n)}(2) / n!$
0	$x^3 + 2x + 1$	13	13
1	$3x^2 + 2$	14	14
2	6x	12	6
3	6	6	1
4	0	0	0
5	0	0	0

Hence, we have that $f(x) = 13 + 14(x-2) + 6(x-2)^2 + (x-2)^3$.

For n > 3, $f^{(n)}(x) \equiv 0$. Hence, from the Lagrange form of the remainder for the Taylor series we have $R_{n+1}(x) = 0$. Consequently, $\lim_{n \to \infty} R_{n+1}(x) = 0$.

2. For $f(x) = \sin x$ and center a = 0, the integral form of the remainder is

$$R_{n+1}(x) = \frac{1}{n}! \int_0^x x^n g_{n+1}(x)\, dx \text{ where } g_n(x) = \begin{cases} \sin x & n \equiv 0 \ (\text{mod} 4) \\ \cos x & n \equiv 1 \ (\text{mod} 4) \\ -\sin x & n \equiv 2 \ (\text{mod} 4) \\ -\cos x & n \equiv 3 \ (\text{mod} 4) \end{cases}.$$

Hence,

$$|R_{n+1}(x)| = |\frac{1}{n}! \int_0^x t^n g_n(t)\, dt| \le \frac{1}{n}! \int_0^{|x|} |t^n g_n(t)|\, dt \le \frac{1}{n}! \int_0^{|x|} t^n\, dt = \frac{|x|^{n+1}}{(n+1)!}.$$

3. From 2 (above) we have, for each (fixed) x, that $\lim_{n \to \infty} R_{n+1}(x) = \lim_{n \to \infty} \frac{|x|^{n+1}}{(n+1)!} = 0$.

4a. Let $f(x) = \log(1+x)$ and center a = 1. Then,

n	$f^{(n)}(x)$	$f^{(n)}(2)$	$f^{(n)}(2) / n!$
0	$\log(1+x)$	$\log 3$	$\log 3$
1	$\frac{1}{1+x}$	$\frac{1}{3}$	$\frac{1}{3}$
2	$\frac{-1}{(1+x)^2}$	$-\frac{1}{9}$	$-\frac{1}{18}$
3	$\frac{2}{(1+x)^3}$	$\frac{2}{27}$	$\frac{1}{81}$
4	$\frac{-6}{(1+x)^4}$	$-\frac{6}{81}$	$-\frac{1}{324}$
5	$\frac{24}{(1+x)^5}$	$\frac{24}{243}$	$\frac{1}{1215}$

Hence, for n = 4 we have $R_{n+1}(x) = \frac{1}{5\,(1+c)^5}(x-3)^5$ for some c between 3 and x.

Section 8.6

1a. If m is an integer, then from (1) we have

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \cdots + x^m = \sum_{k=0}^{m}\binom{m}{k}x^k \quad (*)$$

In (*), let x = 1.

Section 8.7

1a.
$$\lim_{x\to 0^+}\frac{\tan x - x}{x - \sin x} \underset{l'hospital's\,rule}{=} \lim_{x\to 0^+}\frac{\sec^2 x - 1}{1 - \cos x} \underset{l'hospital's\,rule}{=}$$
$$\lim_{x\to 0^+}\frac{2\sec x \sec x \tan x}{\sin x} \underset{algebra}{=} \lim_{x\to 0^+} 2\sec^3 x = 2$$

1b.
$$\lim_{x\to 0^+}\frac{10^x - 5^x}{x} \underset{l'hospital's\,rule}{=} \lim_{x\to 0^+} 10^x \log 10 - 5^x \log 5 = \log 10 - \log 5 = \log 2$$

1c. $$\lim_{x \to 0} \frac{\log \frac{1+x}{1-x}}{x} \underset{l'hospital's rule}{=} \lim_{x \to 0} \frac{2}{1-x^2} = 2$$

2a. $$\lim_{x \to \infty} \frac{\log(1+e^{3x})}{x} \underset{l'hospital's rule}{=} \lim_{x \to \infty} \frac{3e^{3x}}{1+e^{3x}} \underset{algebra}{=} \lim_{x \to \infty} \frac{3}{1+e^{-3x}} = 3$$

2b. $$\lim_{x \to \infty} \frac{\log x}{x^{0.0001}} \underset{l'hospital's rule}{=} \lim_{x \to \infty} \frac{\frac{1}{x}}{0.0001\, x^{-0.9999}} \underset{algebra}{=} \lim_{x \to \infty} \frac{1}{0.0001\, x^{0.0001}} = 0$$

2c. $$\lim_{x \to \infty} x\left(\sqrt{x^2+4} - x\right) \underset{algebra}{=} \lim_{x \to \infty} \frac{x(x^2+4-x^2)}{\sqrt{x^2+4}+x} \underset{algebra}{=} \lim_{x \to \infty} \frac{4x}{\sqrt{x^2+4}+x} \underset{algebra}{=}$$
$$\frac{1}{\lim_{x \to \infty} \frac{\sqrt{x^2+4}+x}{4x}} \underset{algebra}{=} \frac{1}{\lim_{x \to \infty} \frac{\sqrt{x^2+4}}{4x} + \lim_{x \to \infty} \frac{x}{4x}} \underset{algebra}{=} \frac{1}{\lim_{x \to \infty} \sqrt{\frac{x^2+4}{16x^2}} + \frac{1}{4}} \underset{continuity}{=}$$
$$\frac{1}{\sqrt{\lim_{x \to \infty} \frac{x^2+4}{16x^2}} + \frac{1}{4}} \underset{l'hospital's rule}{=} \frac{1}{\frac{1}{4}+\frac{1}{4}} = 2$$

3a. $$\lim_{x \to 1^+} \frac{x - 5x^5 + 4x^6}{(1-x)^2} \underset{l'hospital's rule}{=} \lim_{x \to 1^+} \frac{1 - 25x^4 + 24x^5}{-2(1-x)} \underset{l'hospital's rule}{=}$$
$$\lim_{x \to 1^+} \frac{-100x^3 + 120x^4}{2} = \frac{20}{2} = 10$$

3b. $$\lim_{x \to 1} \frac{1 - 4\sin^2\left(\frac{\pi x}{6}\right)}{1-x^2} \underset{l'hospital's rule}{=} \lim_{x \to 1} \frac{-8\sin\left(\frac{\pi x}{6}\right)\cos\left(\frac{\pi x}{6}\right)\frac{\pi}{6}}{-2x} = \frac{-8\,\frac{1}{2}\,\frac{\sqrt{3}}{2}\,\frac{\pi}{6}}{-2} = \frac{\sqrt{3}\,\pi}{6}$$

4a. $$\lim_{x \to 0} \frac{1}{x} - \frac{1}{\sin x} \underset{algebra}{=} \lim_{x \to 0} \frac{\sin x - x}{x \sin x} \underset{l'hospital's rule}{=}$$
$$\lim_{x \to 0} \frac{\cos x - 1}{x\cos x + \sin x} \underset{l'hospital's rule}{=} \lim_{x \to 0} \frac{-\sin x}{x(-\sin x) + \cos x + \cos x} = 0$$

4b. $$\lim_{x \to \infty} x^{\frac{1}{x}} \underset{algebra}{=} \lim_{x \to \infty} \exp\left(\frac{1}{x}\log x\right) \underset{continuity}{=}$$
$$\exp\left(\lim_{x \to \infty} \frac{\log x}{x}\right) \underset{l'hospital's rule}{=} \exp\left(\lim_{x \to \infty} \frac{1}{x}\right) = 1$$