Solution Set #6

Section 7.10

1a. No. ($\int_0^1 \frac{1}{x^p}\,dx$ diverges if p $\geq$ 1)

1b. Yes. ($\int_0^1 \frac{1}{x^p}\,dx$ converges if p < 1)

1c. Yes. ($\frac{x}{(16-x^4)^{1/3}} \leq \frac{2}{[(4+x^2)(2+x)]^{1/3}} \frac{1}{(2-x)^{1/3}} \leq \frac{2}{8^{1/3}} \frac{1}{(2-x)^{1/3}}$ for x $\in$ [0,2] and $\int_a^b \frac{1}{(b-x)^p}\,dx$ converges if p < 1)

1e. Yes. ($\lim_{x\to 0^+} x^{1/4}\log(1/x) = 0$ which implies there exists a constant A such that $\frac{\log(1/x)}{\sqrt{x}} \leq \frac{A}{x^{3/4}}$ for x $\in$ (0,1] and $\int_0^1 \frac{1}{x^p}\,dx$ converges if p < 1)

1f. Yes. (sin x $\leq$ x for x $\in$ [0,1] which implies that $\frac{\sin x}{x^{3/2}} \leq \frac{1}{x^{1/2}}$ for x $\in$ [0,1] and $\int_0^1 \frac{1}{x^p}\,dx$ converges if p < 1)

3. $\int_0^\infty \frac{x^{s-1}}{1+x}\,dx$ is convergent if and only if both $\int_0^1 \frac{x^{s-1}}{1+x}\,dx$ and $\int_1^\infty \frac{x^{s-1}}{1+x}\,dx$ are convergent. $\int_0^1 \frac{x^{s-1}}{1+x}\,dx$ is convergent if and only if 1- s < 1, i.e., 0 < s. Since $\frac{x^{s-1}}{1+x} \leq \frac{x^{s-1}}{x} = \frac{1}{x^{2-s}}$ for x $\geq$ 1, then $\int_1^\infty \frac{x^{s-1}}{1+x}\,dx$ is convergent if and only if 2 - s > 1, i.e., s < 1. Therefore, $\int_0^\infty \frac{x^{s-1}}{1+x}\,dx$ is convergent if and only if 0 < s < 1.

5a. Yes. (sin t is an odd function; therefore, the C.P.V. $\int_{-\infty}^{\infty} \sin t\,dt = 0$.

5b. No. (|sin t| is an even function and $\int_0^\infty |\sin t|\,dt$ diverges)

5c. Yes. ($1/(1+t^2)$ is an even function and $\int_0^\infty \frac{1}{1+t^2}\,dt$ converges)

7. If f is continuous on [0,1], then f attains a maximum on [0,1], say M. We have then that $\frac{f(x)}{\sqrt{1-x^2}} \leq \frac{M}{\sqrt{1+x}}\frac{1}{\sqrt{1-x}} \leq M\frac{1}{\sqrt{1-x}}$ and $\int_a^b \frac{1}{(b-x)^p}\,dx$ converges if $p < 1$. Therefore, $\int_0^1 \frac{f(x)}{\sqrt{1-x^2}}\,dx$ converges. We have $\int_0^1 \frac{f(x)}{\sqrt{1-x^2}}\,dx = \lim_{b\to 1^-}\int_0^b \frac{f(x)}{\sqrt{1-x^2}}\,dx$. Letting x = sin u, then from Theorem 7.8G we have $\int_0^b \frac{f(x)}{\sqrt{1-x^2}}\,dx = \int_0^{\arcsin(b)} f(\sin(u))\,du$.

Therefore, we have

$$\int_0^1 \frac{f(x)}{\sqrt{1-x^2}}\,dx = \lim_{b\to 1^-}\int_0^b \frac{f(x)}{\sqrt{1-x^2}}\,dx = = \lim_{b\to 1^-}\int_0^{\arcsin(b)} f(\sin(u))\,du = \int_0^{\pi/2} f(\sin(u))\,du.$$

Section 8.1

1a. $\tanh(-x) = \frac{\sinh(-x)}{\cosh(-x)}$ = [by (9) and (10)] $\frac{-\sinh(x)}{\cosh(x)}$ = - tanh(x)

1b. By (8) C'(x) = S(x); hence, C"(x) = S'(x). Then, by (7) S'(x) = C(x). Therefore, C"(x) = C(x).

1c. By S'(x) = C(x) and C(x) > 0 on (-∞, ∞). Theorem 7.7B implies that S(x) is strictly increasing on (-∞, ∞).

1d. Since by (5) S"(x) = S(x) on (-∞, ∞) and since S(0) = 0, then 1c (above) implies that S"(x) > 0 for x > 0 and S"(x) < 0 for x < 0. Hence, S is concave up for x > 0 and S is concave down for x < 0.

Section 8.2

1a. By (14) we have E(x) > 0 and by (15) E'(x) > 0. Thus, Theorem 7.7B implies that E(x) is strictly increasing on (-∞, ∞). By the comments following (14) we have that $\lim_{x\to\infty} E(x) = \infty$. By (12) E(-x) = 1/E(x). Hence, we must have $\lim_{x\to\infty} E(-x) = \lim_{x\to\infty} 1/E(x) = 0$, i.e., $\lim_{x\to-\infty} E(x) = 0$.

1b. By (15) E'(x) = E(x) on (-∞, ∞); hence, E"(x) = E(x) on (-∞, ∞). By (14) E(x) > 0 on (-∞, ∞) which implies E"(x) > 0 on (-∞, ∞). Hence, E(x) is concave on (-∞, ∞).

3.

$$\frac{E(x)+E(-x)}{2} = \frac{(C(x)+S(x))+(C(-x)+S(-x))}{2} =$$
$$\frac{(C(x)+S(x))+(C(x)+-S(x))}{2} = \frac{2C(x)}{2} = C(x)$$

$$\frac{E(x)-E(-x)}{2} = \frac{(C(x)+S(x))-(C(-x)+S(-x))}{2} =$$
$$\frac{(C(x)+S(x))-(C(x)+-S(x))}{2} = \frac{2S(x)}{2} = S(x)$$

4a.

$$\sinh(x)\cosh(y) + \cosh(x)\sinh(y) = \frac{e^{x}-e^{-x}}{2}\,\frac{e^{y}+e^{-y}}{2} + \frac{e^{x}+e^{-x}}{2}\,\frac{e^{y}-e^{-y}}{2} =$$
$$\frac{e^{x+y}+e^{x-y}-e^{-x+y}-e^{-(x+y)}}{4} + \frac{e^{x+y}-e^{x-y}+e^{-x+y}-e^{-(x+y)}}{4} =$$
$$\frac{2e^{x+y}-2e^{-(x+y)}}{4} = \frac{e^{x+y}-e^{-(x+y)}}{2} = \sinh(x+y)$$

4c. By 4a (above) sinh(2x) = sinh(x + x) = sinh(x)cosh(x) + cosh(x)sinh(x) = 2sinh(x)cosh(x).

4d. Using 4b cosh(2x) = cosh(x + x) = cosh(x)cosh(x) + sinh(x)sinh(x) = $\cosh^2 x + \sinh^2 x$.

5a. $$\tanh x = \frac{\sinh x}{\cosh x} = \frac{\frac{e^{x}-e^{-x}}{2}}{\frac{e^{x}+e^{-x}}{2}} = \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$$

5b. Because tanh'(x) = $\operatorname{sech}^2(x) > 0$ on (-∞, ∞), we have that tanh(x) is increasing on (-∞, ∞). We have, from 5a (above) $\lim_{x\to\infty} \tanh x = \lim_{x\to\infty} \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}} = 1$. Similarly, $\lim_{x\to-\infty} \tanh x = -1$. Hence, the range of tanh x is (-1,1).

5c. Note that 1 - $\tanh^2 x = \operatorname{sech}^2 x$. Let y = w(x) be the inverse function to tanh, i.e., let x = tanh y. Then, differentiating we have 1 = (sech^2 y) y'. Solving for y' we have

$$y' = \frac{1}{sech^2 y} = \frac{1}{1-\tanh^2 y} = \frac{1}{1-x^2}.$$

Section 8.3

1a. Applying Theorem 7.8A to the representation for L(x) in (23) we have L'(x) = 1/x for x in $(0,\infty)$. Then, Theorem 7.7B implies that L(x) is increasing for x in $(0,\infty)$.

1b. From 1a (above) we have $L''(x) = -1/x^2 < 0$ for x in $(0,\infty)$. Hence, L is concave down on $(0,\infty)$.

3a. $f(x) = x^a = \exp(a \log x)$. Hence, $f'(x) = \exp(a \log x)\, a\, (1/x) = a\, x^a (1/x) = a\, x^{a-1}$.

3b. $f(x) = a^x = \exp(x \log a)$. Hence, $f'(x) = \exp(x \log a)(\log a) = a^x (\log a)$.

4. By the second mean value theorem for integrals (Theorem 8.5D) we have (for some c between 1 and x (in the case that x > 1)

$$\int_1^x \frac{1}{t^{3/2}}\,dt = \int_1^x \frac{1}{t^{1/2}} \cdot \frac{1}{t}\,dt = \frac{1}{c^{1/2}}\int_1^x \frac{1}{t}\,dt > \frac{1}{x^{1/2}}\int_1^x \frac{1}{t}\,dt = \frac{1}{x^{1/2}}\log x$$

Hence, $\frac{1}{x^{1/2}}\int_1^x \frac{1}{t^{3/2}}\,dt > \frac{\log x}{x}$ (> 0) for x > 1. Hence, integrating the left-hand side we have $\frac{1}{x^{1/2}}\, 2\, \frac{\sqrt{x}-1}{\sqrt{x}} > \frac{\log x}{x} > 0$ for x > 1. But, then the squeeze theorem implies that as x tends to infinity that (log x)/x tends to 0.

Section 8.4

1a. By (35) $\sin(\pi/2 - x) = \sin(\pi/2)\cos(x) - \cos(\pi/2)\sin(x)$. But $\sin(\pi/2) = 1$ and $\cos(\pi/2) = 0$ (see remarks following line (29)). So, $\sin(\pi/2 - x) = \cos(x)$.

1b. By (36) $\cos(\pi/2 - x) = \cos(\pi/2)\cos(x) + \sin(\pi/2)\sin(x)$. But $\sin(\pi/2) = 1$ and $\cos(\pi/2) = 0$ (see remarks following line (29)). So, $\cos(\pi/2 - x) = \sin(x)$.

2a. By (36) $\cos(2x) = \cos(x + x) = \cos(x)\cos(x) - \sin(x)\sin(x) = \cos^2 x - \sin^2 x$. But the later equals, by (34) $(1 - \sin^2 x) - \sin^2 x$. Hence, $\cos(2x) = 1 - 2\sin^2 x$.

3a. By 2a (above) and the remarks following line (29) $0 = \cos(2(\pi/4)) = 1 - 2\sin^2 \pi/4$. Hence, $\sin \pi/4 = \pm\frac{1}{\sqrt{2}}$. Since $\sin x > 0$ for $0 < x < \pi/2$, we have $\sin \pi/4 = \frac{1}{\sqrt{2}}$. Mutatis mutandis, we have $\cos \pi/4 = \frac{1}{\sqrt{2}}$.

4. According to line (28) we have $\sin(x + 2\pi) = \sin((x + \pi) + \pi) = - \sin(x + \pi) = - (- \sin x) = \sin x$. By line (30) and we have $\cos(x) = \sin'(x)$. But the derivative of sine satisfies according to the remarks following line (30) the relation $\sin'(x+\pi) = -\sin'(x)$. Hence, repeating the above argument we have $\cos(x + 2\pi) = \cos(x)$.

7. By the quotient rule

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{\cos x \cos x - \sin x\,(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

Note that $1 + \tan^2 x = \sec^2 x$. Let $y = w(x)$ be the inverse function to tan, i.e., let $x = \tan y$. Then, differentiating we have $1 = (\sec^2 y)\, y'$. Solving for y' we have

$$y' = \frac{1}{\sec^2 y} = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2}\,.$$