Solution Set #5

Section 7.8

1. By Theorem 7.8A $f'(x) = \sqrt{x + x^6}$. Hence, $f'(2) = \sqrt{66}$.

2. $\int_0^3 (3x^2 - 5)\,dx =$ (by Theorem 7.4C)

$\int_0^3 3x^2\,dx + \int_0^3 -5\,dx =$ (by Theorem 7.4B)

$3\int_0^3 x^2\,dx \; -5 \int_0^3 1\,dx =$ (by Theorem 7.8E)

$3\,(\frac{2^3}{3} - 0) \; -5\,(2 - 0)$

4. By Theorem 7.8A *F'*(x) = *f* (x) for x ∈ [a,b]. Since *f* (x) > 0 for x ∈ [a,b], we have *F'*(x) > 0 for x ∈ [a,b]. By Theorem 7.7B which is a corollary of the Mean Value Theorem *F* is strictly increasing for x ∈ [a,b].

5. Let $F(x) = \int_a^x f(t)\,dt$ for x ∈ [a,b]. By Theroem 7.8A, *F'*(x) exists for x ∈ [a,b] (and equals *f* (x)) . Note that *F* (a) = 0. By the Mean Value Theorem there exists a *c* ∈ (a,b) such that $\frac{F(b) - F(a)}{b - a} = F'(c) = f(c)$. Hence, $\int_a^b f(x)\,dx = f(c)\,(b - a)$.

Section 7.9

1a) No. ($\int_1^\infty \frac{1}{x^p}\,dx$ diverges if p ≤ 1).

1b) Yes. ($\int_1^\infty \frac{1}{x^p}\,dx$ converges if p > 1).

1c) No. ($\frac{1}{2x} \le \frac{x}{1+x^2}$ for x ≥ 1 and $\int_1^\infty \frac{1}{x^p}\,dx$ diverges if p ≤ 1).

1d) Yes. ($\frac{43x^2}{1 + 2x^2 + 12x^4} \le \frac{43}{12}\frac{1}{x^2}$ for x ≥ 1 and $\int_1^\infty \frac{1}{x^p}\,dx$ converges if p > 1).

1e) No. ($\int_1^s x\cos(x)\,dx = \cos(s) + s * \sin(s) - \cos(1) - \sin(1) = F(s)$ and $\lim_{s\to\infty} F(s)$ does not exist.)

1f) Yes. ($\frac{1}{(1+x^3)^{\frac{1}{2}}} \le \frac{1}{x^{\frac{3}{2}}}$ for x ≥ 1 and $\int_1^\infty \frac{1}{x^p}\,dx$ converges if p > 1).

1g) No. ($\frac{1}{2x} \le \frac{1}{(1+x^3)^{\frac{1}{3}}}$ for x ≥ 1 and $\int_1^\infty \frac{1}{x^p}\,dx$ diverges if p ≤ 1).

2. Let $F(x) = \frac{1}{2}\frac{1}{(1+x)^2} - \frac{1}{1+x}$ and $G(x) = -\frac{1}{2}\frac{1}{1+x}$. Then, $F'(x) = \frac{x}{(1+x)^3}$ and $G'(x) = \frac{1}{2}\frac{1}{(1+x)^2}$. Hence, $\int_0^s \frac{x}{(1+x)^3}\,dx = F(s) - F(0) = F(s) + \frac{1}{2}$ and $\frac{1}{2}\int_0^s \frac{1}{(1+x)^2}\,dx = G(s) - G(0) = G(s) + \frac{1}{2}$. Since $\lim_{s\to\infty} F(s) = \lim_{s\to\infty} G(s) = 0$, we have $\int_0^\infty \frac{x}{(1+x)^3}\,dx = \frac{1}{2} = \frac{1}{2}\int_0^\infty \frac{1}{(1+x)^2}\,dx$.

4. False. Let $f(x) = \sin(2\pi n x)$ for x ∈ [n,n+1], n ∈ I. Then, f is continuous on [1,∞). {f is obviously piece-wise continuous on [1,∞) and at each integer in [1,∞) f has two one-sided limits which are the same (namely 0)}. Let ε > 0 and let $N = [\frac{1}{\varepsilon}] + 1$. Then, for s ≥ N we have $|\int_s^\infty f(x)\,dx| \le \frac{1}{(N+1)\pi} < \frac{1}{\frac{1}{\varepsilon}\pi} < \varepsilon$. Hence, the integral converges, but $\lim_{x\to\infty} f(x) \ne 0$.

5. Proceeding by contradiction. Suppose $L \ne 0$. WoLog $L > 0$. Since, $\lim_{x\to\infty} f(x) = L$, then there exists N such that for x > N we must have f (x) > L / 2. But then, $\int_N^\infty f(x)\,dx$ diverges which implies that $\int_1^\infty f(x)\,dx$ diverges which is a contradiction.

6. Let $f(x) = |\sin(\pi x)|$. Then, f (n) = 0 for all n ∈ I and hence $\sum_{n=1}^{\infty} f(n)$ converges. But, $\int_1^\infty f(x)\,dx$ diverges .

7. Let $f(x) = \begin{cases} 2(x-n)+\frac{1}{n} & x \in [n-\frac{1}{2n}, n] \\ (-2(x-n)+\frac{1}{n} & x \in [n, n+\frac{1}{2n}] \\ 0 & \text{otherwise} \end{cases}$, for n ∈ I.

Then, $f(n) = \frac{1}{n}$ and hence, $\sum_{n=1}^{\infty} f(n)$ diverges. But, $\int_1^{\infty} f(x)\,dx = \frac{1}{4} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n^2}$ which converges.