Solution Set #4

Section 7.4

1. From theorems 7.4B and 7.4C we have
$$\int_0^1 (2x^2-3x+5)\,dx = 2\int_0^1 x^2\,dx - 3\int_0^1 x\,dx + 5\int_0^1 1\,dx$$
From problems 7.2.2 and 7.2.3 combined with 7.2.7 we have
$$\int_0^1 x\,dx = \lim_{n\to\infty}\frac{n(n+1)}{2n^2} = \frac{1}{2} \quad\text{and}\quad \int_0^1 x^2\,dx = \lim_{n\to\infty}\frac{n(n+1)(2n+1)}{6n^3} = \frac{1}{3}$$
From class $\int_a^b c\,dx = (b-a)*c$ for any constant c. Hence,
$$\int_0^1 (2x^2-3x+5)\,dx = \frac{2}{3} - \frac{3}{2} + 5 = 5\frac{5}{6}$$

4. To show that F is continuous at x in [a,b] requires showing that $\lim_{h\to 0} F(x+h) = F(x)$ which is equivalent to showing
$$\lim_{h\to 0} |F(x+h) - F(x)| = 0. \qquad (*)$$
To show (*), let $\varepsilon > 0$. Since f is continuous on [a,b], we have by theorem 6.6D that f is bounded on [a,b], say by M. Let $\delta = \varepsilon/M$. Claim that for $|h| < \delta$ that $|F(x+h) - F(x)| < \varepsilon$, which will imply that (*) holds.

Verification of the claim: We have
$$|F(x+h) - F(x)| = \left|\int_x^{x+h} f(t)\,dt\right| \le \left|\int_x^{x+h} |f(t)|\,dt\right|$$
by theorems 7.4F (and 7.4G in the case where h < 0). But, then
$$\left|\int_x^{x+h} |f(t)|\,dt\right| \le M|h|$$
and the claim follows.

5. a) If $0 \le x \le 1$, then $1 \le x+1 \le 2$ and subsequently, $1 \le \sqrt{1+x} \le \sqrt{2}$. Hence,
$$x^2 \ge \frac{x^2}{\sqrt{1+x}} \ge \frac{x^2}{\sqrt{2}}.$$

b) From a) and theorem 7.4E we have $\int_0^1 x^2\,dx \ge \int_0^1 \frac{x^2}{\sqrt{1+x}}\,dx \ge \int_0^1 \frac{x^2}{\sqrt{2}}\,dx$. But, then from the comments in problem 1 we have $\int_0^1 x^2\,dx = \frac{1}{3}$. Hence,
$$\frac{1}{3} \ge \int_0^1 \frac{x^2}{\sqrt{1+x}}\,dx \ge \frac{1}{3\sqrt{2}}$$

7. If f is continuous at c in [a,b] and $f(c) > 0$, then there exists $\delta > 0$ such on the interval $J = B(c, \delta) \cap [a,b]$ we have $f(x) > f(c)/2 > 0$. Furthermore, note that $|J| \geq \delta > 0$. Let χ be the characteristic function of the interval J and let $g(x) = \frac{f(c)}{2}\chi(x)$. Then, on [a,b] we have $f(x) \geq g(x)$. Hence, by theorem 7.4E we have $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$. But, $\int_a^b g(x)\,dx = \frac{f(c)}{2}|J| > 0$.

8. Suppose to the contrary that f is not identically 0 on [a,b]. There, there exists a c in [a,b] such that at c we have $f(c) > 0$. Then, by problem 7 we have $\int_a^b f(x)\,dx > 0$, which contradicts the hypotheses of the problem.

Section 7.5

1. Suppose f is the constant function $f(x) = c$. Then, for any x in [a,b] we have that

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0}\frac{c-c}{h} = 0.$$

2. The hypothesis that $f'(c)$ exists means that $\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$ exists. But, then the limit theorems imply that $b\lim_{h\to 0}\frac{f(c+h)-f(c)}{h} = \lim_{h\to 0}\frac{b\,f(c+h)-b\,f(c)}{h}$. But, since $g(x) = bf(x)$ on an interval containing c, we have that $g'(c)$ exists and equals $bf'(c)$.

5. a) Let m = -n. Then $x^n = 1/x^m$. If $x \neq 0$, then theorem 7.5C (quotient rule) implies that $(x^n)' = (\frac{1}{x^m})' = \frac{-mx^{m-1}}{x^{2m}} = \frac{-m}{x^{m+1}}$. But the latter term is nx^{n-1}.

7. Since $f'(c)$ exists and since $a < c < b$, then the lemma on page 195 implies that there exists and interval $(0,\delta)$ so that for all $h \in (0,\delta)$ we can write

$$f(c+h) = f(c) + h\,F(h) \quad (**)$$

where F is continuous at $h = 0$ and $F(0) = f'(c) > 0$.

Since F is continuous at $h = 0$ and since $F(0) > 0$, there exists an interval $(0,\delta_1)$ on which $F(x) > 0$. Wolog we may take $\delta_1 < \delta$. Then, for all $h \in (0,\delta_1)$ we have $h\,F(h) > 0$ and hence, from (**) $f(c+h) > f(c)$.

9. False. Consider $f(x)=\begin{cases} 2x+x^2\sin\dfrac{1}{x} & , \ x\neq 0 \\ 0 & , \ x=0 \end{cases}$. Then, f is defined for all real x and f' exists for all real x and $f'(0) > 0$, but f is not strictly increasing on any open interval which contains 0.

Section 7.6

1. Suppose to the contrary that there exists a k such that $f(x) = x^3 - 3x + k$ has two distinct roots in [0,1], say x_1 and x_2 with $0 \le x_1 < x_2 \le 1$. Then, by Rolle's theorem we have that $f'(x) = 3x^2 - 3$ has a root at some c in $(x_1, x_2) \subset (0,1)$. But, the roots of f' are ± 1 which is a contradiction.

2. a) $f(x) = \sin x$ on $[0,\pi]$. Yes. f is continuous on $[0,\pi]$ and differentiable on $(0,\pi)$ and $f(0) = 0 = f(\pi)$.

   b) $f(x) = \sqrt{x}\,(x-1)$ on [0,1]. Yes. f is continuous on [0,1] and differentiable on (0,1) and $f(0) = 0 = f(1)$.

   c) $f(x) = \sin 1/x$ on $[-1/\pi, 1/\pi]$, $x \neq 0$; $f(0) = 0$. No. f is not differentiable at $x = 0$ and hence not on $(-1/\pi, 1/\pi)$.

   d) $f(x) = x^2$ on [0,1]. No, $f(0) \neq f(1)$.

3. For $f(x) = (x - a)(b - x)$ we have $f'(x) = a+b - 2x$. So, $c = (a+b)/2$ satisfies $f'(c) = 0$ and $c \in (a,b)$.

Section 7.7

1. a) $f(x) = x/(x-1)$ on [0,2]. No. f is not defined at $x = 1$ and hence cannot be continuous or differentiable at $x = 2$.

   b) $f(x) = x/(x-1)$ on [2,4]. Yes. f is continuous on [2,4] and differentiable on (2,4). $c = 1 + \sqrt{3}$ satisfies the conclusion of the mean value theorem.

   c) $f(x) = Ax + B$ on [a,b]. Yes. f is continuous on [a,b] and differentiable on (a,b). Any $c \in (a,b)$ satisfies the conclusion of the mean value theorem.

   d) $f(x) = 1 - x^{2/3}$ on [-1,1]. No. f is not differentiable at $x= 0$ and hence, not on (-1,1).

2. a) $c = 1/2$

b) $c = -\pi / 4$

3. If $f'(x)$ and $g'(x)$ exist for all x in [a,b], then both f and g are both continuous on [a,b]. Furthermore, $g'(x) \neq 0$ for all x in [a,b] implies that $g(a) \neq g(b)$ — otherwise, Rolle's would be contradicted. Finally, since $g'(x) \neq 0$ for all x in [a,b], then we cannot have both $f'(t) = g'(t) = 0$ for any t in (a,b).

4. Suppose $f'(x) = 0$ for all x in (a,b). Let c be a specific (fixed) point in (a,b). Then, for any other point x in (a,b), $x \neq c$, we have by the mean value theorem that

$$\frac{f(x) - f(c)}{x - c} = 0 \quad (***)$$

But, (***) implies that $f(x) = f(c)$. Therefore, f is constant on (a,b).