Solution Set #3

Section 7.1

1. Suppose that $A \setminus B$ were a set of measure zero. Then, the union of two sets of measure set, $A \setminus B \cup B$ would again be a set of measure zero. But the union would be A which was given as not a set of measure zero.

2. (a) Suppose that [a,b] could be covered by a finite union of intervals $I_1, I_2, \ldots, I_n$ such that $|I_k| = b_k - a_k$, k = 1, 2, . . . , n and the sum of the lengths $\sum_{k=1}^{n} b_k - a_k < b-a$. Wolog we may suppose that no interval $I_k \subset I_j$ for $k \neq j$ and that the intervals I_k are ordered by their left endpoints, i.e., $a_1 < a_2 < \ldots < a_n$. Furthermore, we may assume that no interval $I_k \subset I_{k-1} \cup I_{k+1}$, k = 2, 3, . . . , n-1. Therefore, we must have that $a \in I_1$. If $x \in I_1$, then $x-a < b_1 - a_1$. In particular, $a_2 \in I_1$. If $x \in I_2$, then $x - a_2 < b_2 - a_2$. Therefore, combining the above two facts $x - a < (x - a_2) + (a_2 - a) < (b_2 - a_2) + (b_1 - a_1) < b - a$. Proceeding inductively we have that if $x \in I_k$, then $x - a_k < b_k - a_k$ and

(*)
$$\begin{aligned} x - a &< (x - a_k) + (a_k - a_{k-1}) + (a_{k-1} - a_{k-2}) + \ldots + (a_3 - a_2) + (a_2 - a) \\ &< (b_k - a_k) + (b_{k-1} - a_{k-1}) + (b_{k-2} - a_{k-2}) + \ldots + (b_2 - a_2) + (b_1 - a_1) < b - a. \end{aligned}$$

Specifically, b must belong to one of the intervals since the union of the intervals covers [a,b]. But, then by (*) we would have $b - a < b - a$.

(b) Suppose that [a,b] were of measure zero. Then, given any $\varepsilon > 0$ (in particular, $\varepsilon = (b - a)/2$) there would exist a collection of intervals $\{ I_\alpha \}$, $\alpha \in A$, such that [a,b] could be covered by $\bigcup_{\alpha \in A} I_\alpha$ and $\sum_{\alpha \in A} |I_\alpha| < \varepsilon$. But since [a,b] is compact, then there would exist a finite subcollection which would cover [a,b] for which the sum of the lengths would be even smaller. But, this latter assertion is impossible by part (a) of the problem.

3. This is a direct consequence of problems 1 and 2 above with A = [a,b] and B = {a,b}.

4. (a) Since the rationals are a countable set, then corollary 7.1C applies and asserts that the rationals are a set of measure zero.

(b) This is a direct consequence of problems 1 and 4(a) above with $A = \mathbf{R}^1$ and $B = \mathbf{Q}$, where **Q** is the set of rationals.

5. False. Let $f(x) = 0$ on [0,1]. Let $g(x) = \begin{cases} 0 & x \text{ is irrational} \\ 1 & x \text{ is rational} \end{cases}$, $x \in [0,1]$. Then $f \equiv g$ on [0,1] a.e., but g is not continuous anywhere on [0,1].

Section 7.2

1. $U[f, \sigma] = \frac{1}{3}\cdot\frac{1}{3} + \frac{2}{3}\cdot\frac{1}{3} + 1\cdot\frac{1}{3} = \frac{2}{3}$

$L[f, \sigma] = 0\cdot\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{3} + \frac{2}{3}\cdot\frac{1}{3} = \frac{1}{3}$

2. $U[f, \sigma] = \frac{1}{n}\cdot\frac{1}{n} + \frac{2}{n}\cdot\frac{1}{n} + \cdots + \frac{n}{n}\cdot\frac{1}{n} = \frac{1}{n^2}\sum_{k=1}^{n} k = \frac{\frac{n(n+1)}{2}}{n^2}$

$\lim_{n\to\infty} U[f, \sigma] = \frac{1}{2}$

6. (a) Since f is continuous on [a,b] which is compact, then by theorem 6.8C f is uniformly continuous on [a,b].

(b) Since f is uniformly continuous on [a,b], then by the definition of uniform continuity given $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - f(y)| < \frac{\varepsilon}{b-a} \quad \text{whenever} \quad |x - y| < \delta.$$

(c) Choose $n \geq [[\frac{b-a}{\delta}]] + 1$ and let σ be the subdivision of [a,b] given by $\sigma = \{ a, a+1/n, a+2/n, \ldots, a+(n-1)/n, b\}$. Then, each component interval I_k satisfies $|I_k| = 1/n$. Hence, for any $x,y \in I_k$ we have $|x - y| < \delta$ and by (b) we have $|f(x) - f(y)| < \varepsilon/(b-a)$. Hence,

$$M[f, I_k] - m[f, I_k] < \varepsilon/(b-a). \qquad (*)$$

(d) Since (*) holds for each k, then if we sum (**) over k we have

$$U[f, \sigma] - L[f, \sigma] = \sum_{k=1}^{n} (M[f, I_k] - m[f, I_k])\,|I_k| < \sum_{k=1}^{n} \frac{\varepsilon}{b-a}|I_k| = \varepsilon \quad (**)$$

(e) Since for each $\varepsilon > 0$ given in (b) there exists a subdivision σ given in (c) so that (**) holds, then by theorem 7.2G we have the $f \in \mathscr{R}$ [a,b].

7. (a) Consider the component interval I_k. Since f is continuous on I_k, then by theorem 6.6F we have f obtains a maximum on I_k, i.e., there exists $x_{max} \in I_k$ such that $M[f, I_k] = f(x_{max})$, and we also have that f obtains a minimum on I_k, i.e., there exists $x_{min} \in I_k$ such that $m[f, I_k] = f(x_{min})$. Hence, for any $x_k^* \in I_k$ we must have

$$m[f, I_k] \leq f(x_k^*) \leq M[f, I_k] \quad (*)$$

If we multiply (*) by $|I_k|$ and sum over k we obtain

$$L[f, \sigma_n] \leq \frac{1}{n}\sum_{k=1}^{n} f(x_k^*) \leq U[f, \sigma_n]. \quad (**)$$

(b) Let $\varepsilon > 0$. By the construction in problem 6 above, if n is choose sufficiently large then σ_n is the partition described in 6(c) and from 6(d) we have

$U[f, \sigma_n] - L[f, \sigma_n] < \varepsilon$. Since $L[f, \sigma_n] < \int_a^b f$ we have that $U[f, \sigma_n] - \int_a^b f < \varepsilon$. Since, on the other hand we always have $0 \le U[f, \sigma_n] - \int_a^b f$. Then, we can conclude that $\lim_{n \to \infty} U[f, \sigma_n] = \int_a^b f$. Similarly, since $\int_a^b f < U[f, \sigma_n]$ we have that $\int_a^b f - L[f, \sigma_n] < \varepsilon$. Likewise, we always have $0 \le \int_a^b f - L[f, \sigma_n]$. Hence, $\lim_{n \to \infty} L[f, \sigma_n] = \int_a^b f$. But, since both $\lim_{n \to \infty} U[f, \sigma_n] = \int_a^b f$ and $\lim_{n \to \infty} L[f, \sigma_n] = \int_a^b f$, then (**) and the sandwich (squeeze) theorem imply that $\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} f(x_k^*) = \int_a^b f$.

9. Note: by problem 7, for f continuous on [a,b] we have $\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} f(x_k^*) = \int_a^b f$

(a) Choose $f(x) = x^2$. Then, by problem 7, the indicated limit equals $\int_a^b f$ for the interval [0,1], which equals (from Calc I) 1/3.

(b) Choose $f(x) = \sin \pi x$. Then, by problem 7, the indicated limit equals $\int_a^b f$ for the interval [0,1], which equals (from Calc I) $2/\pi$.

(c) Choose $f(x) = e^{3x}$ Then, by problem 7, the indicated limit equals $\int_a^b f$ for the interval [0,1], which equals (from Calc I) $(e^3 - 1)/3$.

Section 7.3

1. (a) Yes. f is continuous except on the set of points {0, 1/10, 2/10, .. 1} which is a finite set, and, hence, of measure zero.

(b) Yes. f is continuous except on the set {0}, which is a finite set, and, hence, of measure zero.

(c) Yes. f is continuous except on the set rationals which is a countable set, and, hence, of measure zero.

(d) No. f is then not continuous anywhere on [0,1].

2. (a) $\omega[f,x] = 0$ for x not in {0, 1/10, 2/10, .. 1} and $\omega[f,x] = 1$ for x in {0, 1/10, 2/10, .. 1}.

If x is not in {0, 1/10, 2/10, .. 1} then there exists an open interval I_x which contains x and does not intersect {0, 1/10, 2/10, .. 1}. Hence, on I_x we have $f(x) \equiv 0$, which implies that $\omega[f,I_x] = 0$. Since $\omega[f,x] \le \omega[f,I_x]$, then $\omega[f,x] = 0$.

If x is in {0, 1/10, 2/10, .. 1}, then for any open interval J which contains x we would have $\omega[f,J] = 1$. Hence, $\omega[f,x] = 1$.

(b) $\omega[f,x] = 0$ for $x > 0$ and $\omega[f,0] = 8$.

If x is not 0, then f is continuous at x and hence, $\omega[f,x] = 0$. If $x = 0$, then on any open interval J which contains 0, we have that there exists an integer k such $x_k = [(4k-1)\pi/2]^{-1} \in J$ which implies that $f(x_k) = -1$. Since, $f(0) = 7$, then $\omega[f,J] = 8$. Hence, $\omega[f,0] = 8$.

3. If $f \in \mathscr{R}[a,b]$, then by theorem 7.3A f is continuous a.e. Let $E = \{ x \in [a,b] : f$ is discontinuous at $x \}$. Let $E_1 \ \{ x \in [a,b] : |f|$ is discontinuous at $x \}$ Suppose that $y \in [a,b] \setminus E$. Then, by problem 5.1.4 we have that $|f|$ is continuous at y also. Therefore, the $E_1 \subset E$. Hence, E_1 is also a set of measure zero and $|f| \in \mathscr{R}[a,b]$.

4. False. The example in problem 7.1.5 illustrates the non-validity of the claim.

5. True. . Let $E = \{ x \in [a,b] : f$ is discontinuous at $x \}$. Then, E is a set of measure zero. Let $E_1 = \{ x \in [a,b] : g$ is discontinuous at $x \}$. Suppose $f = g$ on [a,b] except for on a finite point set, say $S = \{x_1, x_2, \ldots, x_n\}$. Then $E_1 \subset E \cup S$ because if $y \in [a,b] \setminus (E \cup S)$, then f is continuous at y and g agrees with f on a open interval I_y containing y which implies that g is also continuous at y. However, $E \cup S$ is a set of measure zero since both E and S are sets of measure zero. Hence, E_1 is a set of measure zero and by theroem 7.3A $g \in \mathscr{R}[a,b]$.