Solution Set #2

Section 6.4

2. a. Consider the sequence with terms given by $s_n = 1/n$. $\{s_n\}$ is Cauchy in (0,1), but does not converge in (0,1). Hence, (0,1) is not complete.

   b. Let $\{s_n\}$ be any Cauchy sequence in (0,1) with the discrete metric d. Since $\{s_n\}$ is Cauchy, then for $\varepsilon = 1/2$ there exists N such for n, $m \geq N$ we have $d(s_n,s_m) < 1/2$. But $d(s_n,s_m) < 1/2$ implies that $d(s_n,s_m) = 0$ which in turn implies that $s_n = s_m$. Hence for $n \geq N$ the terms s_n are all constant, i.e., $s_n = s_N$. Therefore the sequence converges (to s_N). Since the sequence $\{s_n\}$ was an arbitrary Cauchy sequence in (0,1) with the discrete metric, we have (0,1) with the discrete metric is complete.

3. Let $\{p_n\}$ be a Cauchy sequence in $\mathbf{R}^2$, $p_n = \langle x_n,y_n \rangle$. In the verification of problem 4.3.2 that it was shown that (by the triangle inequality) that $| x_n - x_m | \leq | p_n - p_m |$ (where the first absolute value is taken on points x_k in $\mathbf{R}^1$ and the second absolute value is taken on points p_k in $\mathbf{R}^2$) and $| y_n - y_m | \leq | p_n - p_m |$ (where the first absolute value is taken on points y_k in $\mathbf{R}^1$ and the second absolute value is taken on points p_k in $\mathbf{R}^2$). Hence, if $\{p_n\}$ is Cauchy in $\mathbf{R}^2$, then so are $\{x_n\}$ and $\{y_n\}$ in $\mathbf{R}^1$. But, $\mathbf{R}^1$ is complete; therefore, the sequences $\{x_n\}$ and $\{y_n\}$ both converge. Hence, but the second part of problem 4.3.2, the sequence $\{p_n\}$ converges. Since the sequence $\{p_n\}$ was an arbitrary Cauchy sequence in $\mathbf{R}^2$, we have $\mathbf{R}^2$ is complete.

5. Let x, $y \in (0,1/3]$. Then $|T(x) - T(y)| = |x^2 - y^2| = |(x-y)(x+y)| \leq (2/3)|x-y|$. Hence, using $\alpha = 2/3$, we see that T is a contraction on (0,1/3]. A similar argument would show that T is a contraction on [0,1/3] which is complete. By theorem 6.4F there exists a unique fixed point for T on [0,1/3]. 0 satisfies the functional equation Tx = x on [0,1/3] so 0 is the unique fixed point. But $0 \notin (0,1/3]$, so T has no fixed point on (0,1/3].

7. Since M is totally bounded, by theorem 6.3H every sequence in M has a Cauchy subsequence. Since M is complete, each such Cauchy subsequence must be convergent to a point in M.

Section 6.5

2. Let $S = \{x_1, x_2, \ldots, x_n\}$ be a finite subset of a metric space M. Let $\{G_\alpha\}$, $\alpha \in A$, be an open cover of S. Then, for each x_k choose an index $k \in A$, such that $x_k \in G_k$. Then by construction $\{G_k\}$ is a finite subcover of S. Since $\{G_\alpha\}$ was arbitrary, by theorem 6.5H M is compact.

3. a. Let S be a compact subset of $\mathbf{R}^2$. By theorem 6.5D, S must be closed. Suppose S were not bounded. Then, we could find a sequence $\{s_n\}$ in S such that for each n we would have $|s_n| > n$. But, this sequence could have no convergent subsequence. That would contradict theorem 6.5B; hence, S must be bounded.

b. Suppose we have a closed bounded subset of $\mathbf{R}^2$. By problem 6.3.1 any bounded subset of $\mathbf{R}^2$ is totally bounded. By problem 6.4.3 $\mathbf{R}^2$ is complete and by theorem 6.4C any closed subset of $\mathbf{R}^2$ is also complete. Hence, by definition 6.5A any closed bounded subset of $\mathbf{R}^2$ is compact.

4. Let A and B be compact subsets of $\mathbf{R}^1$. By theorem 6.5D both A and B are closed subsets of $\mathbf{R}^1$. By problem 5.5.12 A×B is a closed subset of $\mathbf{R}^2$. By definition 6.5A both A and B are totally bounded, which implies, in particular, that they are bounded. The triangle inequality implies that if A and B are bounded, then A×B is also bounded. Hence, by problem 6.5.3, we have A×B is compact.

6. Suppose to the contrary there existed a finite subcover of (0,1), say $\{ I_{x_1}, I_{x_2}, \ldots, I_{x_n} \}$. Then, since $\{x_1, x_2, \ldots, x_n\}$ is a finite set there exists a minimum for it say x_{min}. But, then for for $0 < y < x_{min}/2$ we would have that $y \notin I_{x_k}$ for any k, contradicting the fact that $\{ I_{x_1}, I_{x_2}, \ldots, I_{x_n} \}$ was to have been a finite subcover of (0,1).

Section 6.6

2. Clearly, for any $x \in \mathbf{R}^1$ we have $1 = \frac{1}{1+0^2} \geq \frac{1}{1+x^2}$, so f attains a maximum value (at x = 0). On the other hand, for any $x \in \mathbf{R}^1$ we have if $y \geq |x|$ then $\frac{1}{1+y^2} < \frac{1}{1+x^2}$, so that f does not attain a minimum value.

3. Let $f(x) = \arctan x$. Then, f is bounded and strictly increasing on $\mathbf{R}^1$, which implies that f attains neither a maximum nor a minimum on $\mathbf{R}^1$.

4. Let $f(x) = x$ on [0,1). Then at x=0, f attains a minimum value, but for any x in [0,1) for $y = (x+1)/2$ we have $y > x$ and hence $f(y) > f(x)$.

5. By theorem 6.6F there exists a x_{max} such that f attains a maximum on M at x_{max} and there exists a x_{min} such that f attains a minimum on M at x_{min} Let $f(x_{max}) = d$ and $f(x_{min}) = c$. By theorem 6.2D the range of f is connected, since the domain M is connected by hypothesis. Since range of f is connected and contains c and d, then the interval [c,d] belongs to the range of f. Hence, every value e between c and d belongs to the range of f.

Section 6.7

1. a. Yes, f is continuous because f sends nearby points on the flat map to nearby points on the globe.

b. No, f^{-1} is not continuous, because the globe would have to be cut along someline to map it to a flat map and points nearby to each other but on opposite sides of the line would not be sent to image points nearby each other on the flat map.

2. Let n be a fixed positive integer. From results last semester $f(x) = x^n$ is continuous on $\mathbf{R}^1$. Let $N > 0$ be fixed. Then, f is continuous on the restricted domain $[0,N]$ and one-to-one on $[0,N]$. Since $[0,N]$ is compact, then by theorem 6.7B we have f^{-1} is continuous on its domain $[0,N^n]$. Since, N was arbitrary, we can conclude that f^{-1} is continuous on $[0,\infty)$.

Section 6.8

2. If the absolute value of the secant line for arbitrary points x and y in [a,b] is bounded by 1, then we have that $\frac{|f(x) - f(x)|}{|x - y|} \leq 1$ for arbitrary x and y in [a,b]. But, that implies that $|f(x) - f(y)| \leq |x - y|$ for arbitrary x and y in [a,b]. Given $\varepsilon > 0$, then choose, for the requirement of uniform continuity, $\delta = \varepsilon$. Then $|x - y| < \delta$ will imply that $|f(x) - f(y)| < \varepsilon$ and uniform continuity thus holds.

3. a. Yes. f is continuous on [0,1] and [0,1] is compact so theorem 6.8C applies.
   b. No. f is continuous on $[0,\infty)$, but $[0,\infty)$ is not compact, so theorem 6.8C does not apply. Furthermore, for $\varepsilon = 1$, we can see that no δ can be found so that $|f(x) - f(y)| < 1$ for $|x - y| < \delta$ for all x and y in $[0,\infty)$. Specfically, choose x_0 so that $(x_0 + \delta)^3 > x_0^3 + 2$. Then on the interval $[x_0, x_0+\delta]$ the function x^3 which change by 2. Hence, there will exist a, b in the interval $[x_0, x_0+\delta]$ so that ($|a - b| < \delta$ and) $|a^3 - b^3| \geq 1$.
   c. No. f is continuous on $[0,\infty)$, but $[0,\infty)$ is not compact, so theorem 6.8C does not apply. Futhermore, Furthermore, for $\varepsilon = 1$, we can see that no δ can be found so that $|f(x) - f(y)| < 1$ for $|x - y| < \delta$ for all x and y in $[0,\infty)$. Specfically, choose x_0 so that $(x_0 + \delta)^2 > x_0^2 + 2\pi$. Then, on the interval $[x_0, x_0+\delta]$ the function $\sin x^2$ which go through a full period. Hence, there will exist a, b in the interval $[x_0, x_0+\delta]$ so that ($|a - b| < \delta$ and) $|\sin a^2 - \sin b^2| \geq 1$.
   d. Yes. f is continuous on $[0,\infty)$. $[0,\infty)$ is not compact so theorem 6.8C does not apply. But, on $[0,\infty)$ the absolute value of the secant line for arbitrary points x and y in $[0,\infty)$ is bounded in absolute value by 1. Hence, an extension of problem 6.8.2 will give us that f is uniformly continuous on $[0,\infty)$.

5. Let $\varepsilon > 0$ be given. Since $\lim_{x\to\infty} f(x) = 0$, there exists a N so that for $x \geq N$ we must have $|f(x)| < \varepsilon/2$. Then, for arbitrary $x, y \geq N$ we have that $|f(x) - f(y)| < \varepsilon$. Similarly, there exists $M > 0$ so that for arbitrary $x, y \leq -M$ we have that $|f(x) - f(y)| < \varepsilon$. On the compact interval $[-M-1, N+1]$ theorem 6.8D implies f is uniformly continuous, i.e., for this ε there exists a δ (which we may suppose is less than 1) so that for x and y in $[-M-1, N+1]$ with $|x - y| < \delta$ we have $|f(x) - f(y)| < \varepsilon$. We have now for any x and y in $\mathbf{R}^1$ with $|x - y| < \delta$ that $|f(x) - f(y)| < \varepsilon$. Since ε was arbitrary, f is uniformly continuous on $\mathbf{R}^1$.

8. Let $f : \langle M_1, \rho_1\rangle \to \langle M_2, \rho_2\rangle$ be uniformly continuous on M_1. Let $\{x_n\}$ be a Cauchy sequence in M_1. Let $\varepsilon > 0$ be given. Then, there exists a $\delta > 0$ such that if x, y (in M_1)

satisfy $\rho_1 (x,y) < \delta$, then $\rho_2(f(x), f(y)) < \varepsilon$. Since $\{x_n\}$ is Cauchy, there exists a N such that n, m $\geq$ N implies that $\rho_1 (x_n , x_m) < \delta$. Therefore, for n, m $\geq$ N we have that $\rho_2(f(x_n), f(x_m)) < \varepsilon$. Since ε was arbitrary, the sequence $\{f(x_n)\}$ is Cauchy.