Solution Set #10

Section10.4

1. Let $\varepsilon = 1$ (any number less than 2). For any $\delta > 0$ there exists N such that $\pi/N < \delta$. Consequently for x = 0 and y = π/N we will have $|f_N(x) - f_N(y)| = 2 > \varepsilon$ and $|x - y| < \delta$.

2. It suffices to show that the sequence $\{\varphi_n\}$ is equicontinuous. Let $\varepsilon > 0$. Claim we can choose $\delta = \varepsilon/M$ to verify the condition for equicontinuity. Specifically, for x, y in [a,b] from the mean value theorem we have $|\varphi_n(x) - \varphi_n(y)| = |\varphi'(c)|\,|x - y|$ for some c between x and y. But, then by hypothesis $|\varphi'(c)| \le M$. If $|x - y| < \delta$, then we have $|\varphi_n(x) - \varphi_n(y)| < \varepsilon$.

4. Let $f_n(x) \equiv n$ on [0,1]. Then, $\{f_n\}$ is clearly equicontinuous, but $\{f_n\}$ has no convergent subsequence.

Section 11.1

1. Let $G_1 = \bigcup_{n\in A} I_n$ and $G_2 = \bigcup_{m\in B} J_m$ where each indexing set A and B is either finite or countable and the I_n and J_m are open intervals and the intervals $\{I_n\}$ are pariwise disjoint and the intervals $\{J_m\}$ are pariwise disjoint. For each m = 1, 2, 3, . . . let $N_m = \{n \in A: I_n \subset J_m\}$. Then, each $n \in A$ belongs to some N_m for some $m \in B$. Since the $\{I_n\}$ are pairwise disjoint, then we have

$$\left|\bigcup_{n\in N_m} I_n\right| = \sum_{n\in N_m} |I_n| \le |J_m| \quad (*)$$

When we sum (*) over the $m \in B$ we obtain on the right side that $\sum_{m\in B} |J_m| = |G_2|$. On the other hand, when we sum (*) over the $m \in B$ we obtain on the left side that $\sum_{m\in B} \left|\bigcup_{n\in N_m} I_n\right| = |G_1|$ since every I_n belongs to a unique J_m. Hence, we have $|G_1| \le |G_2|$.

2. Let χ_j be the characteristic function of I_j for j = 1, 2, . . . k. Let χ be the characteristic function of $I_1 \cup I_2 \cup \ldots \cup I_k$. Then for each x in [a,b], we have that

$$\chi(x) \le \chi_1(x) + \chi_2(x) + \ldots + \chi_k(x)$$

Clearly, each $\chi_j \in \mathcal{R}[a,b]$. Since χ is a finite sum of elements in $\mathcal{R}[a,b]$, then $\chi \in \mathcal{R}[a,b]$. Hence, by Theorem 7.4E $\int_a^b \chi \le \int_a^b \chi_1 + \chi_2 + \cdots + \chi_k$. But, then $\int_a^b \chi = |I_1 \cup I_2 \cup \cdots \cup I_k|$ and

$$\int_a^b \chi_1 + \chi_2 + \cdots + \chi_k = \int_a^b \chi_1 + \int_a^b \chi_2 + \cdots + \int_a^b \chi_k = |I_1| + |I_2| + \cdots + |I_k|$$

3. True. If G is open and non-empty, then there exists $x \in G$ and hence there exist an open subinterval (c,d) which contains x and for which $(c,d) \subset G$. That would imply that $|G| \geq d - c > 0$. Hence, $G = \varnothing$.

4. False. Let $F = \{x\}$, where $x \in [a,b]$.

5. By paragraph 3 on page 305, the complement of the Cantor set (in [0,1]) is a union of disjoint open intervals whose lengths sum to 1. Hence, the length of the Cantor sets is $|[0,1]| - 1 = 0$.

Section 11.2

2. Let $\varepsilon > 0$. Then $G_1 = (c-\varepsilon/4,d)$ is an open set which contains [c,d). Also, $G_2 = [a,c) \cup (d-\varepsilon/4,b]$ is an open set which contains $[a,b] \setminus [c,d)$. But, from the construction we have that $|G_1 \cap G_2| = \varepsilon/2 < \varepsilon$. Hence, by Theorem 11.2F we have [c,d) is measurable.

3. By Theorem 11.2C $0 \leq \underline{m}E \leq \overline{m}E$. If $\overline{m}E = 0$, then we have $0 \leq \underline{m}E \leq \overline{m}E \leq 0$, which implies that the outer measure and the inner measure of E are equal to each other and each equal to 0.