Solution Set #1

Section 6.1

1. Example 1. Let $M = \mathbf{R}^1$ and let $A = (0,1)$. Then every open subset O of A can be written as $O = A \cap B$ for B some open subset of M. But since A is open in M, then $A \cap B$ is open in M.

   Example 2. Let $M = \mathbf{R}_d$ and let A be any subset of M.

2.

		[0,1]	$\mathbf{R}^1$	$\mathbf{R}^2$
a)	(1/2,1]	open	not	not
b)	(1/2,1)	open	open	not
c)	[1/2,1)	not	not	not

3. The only subset of $\mathbf{R}^1$ which is open in $\mathbf{R}^2$ is the empty set.

Section 6.2

1. Suppose that f is non-constant. Then there exist $c, d \in f(\mathbf{R}^1)$, such that $c \neq d$. Wolog $c < d$. By corollary 6.2E f takes on every value between c and d. But the interval [c,d] is uncountable.

2. Suppose such an f existed. Let $[a,b] \subset \mathbf{R}^1$, $a < b$. Let $[a,b]_Q = [a,b] \cap \mathbf{Q}$ and $[a,b]_J = [a,b] \cap \mathbf{J}$, i.e., $[a,b]_Q$ is the subset of [a,b] of rationals and $[a,b]_J$ is the subset of [a,b] of irrationals. Since $[a,b]_Q$ is a subset of $\mathbf{Q}$, then $[a,b]_Q$ is countable and $f([a,b]_Q)$ is therefore countable. By hypothesis $f([a,b]_J)$ is a subset of $\mathbf{Q}$ and therefore countable. Hence $f([a,b]) = f([a,b]_Q) \cup f([a,b]_J)$, is countable. But by corollary 62.E $f([a,b])$ must contain the interval between $f(a)$ and $f(b)$, which is uncountable. This is a contradiction.

3. Let $A = [0,1/2)$ and $B = [1/2,1]$. Then since every subset of $\mathbf{R}_d$ is open, we have both A and B are open. Together they form a non-trivial disjoint open decomposition of [0,1]. By 6.2A part (b), [0,1] is not connected.

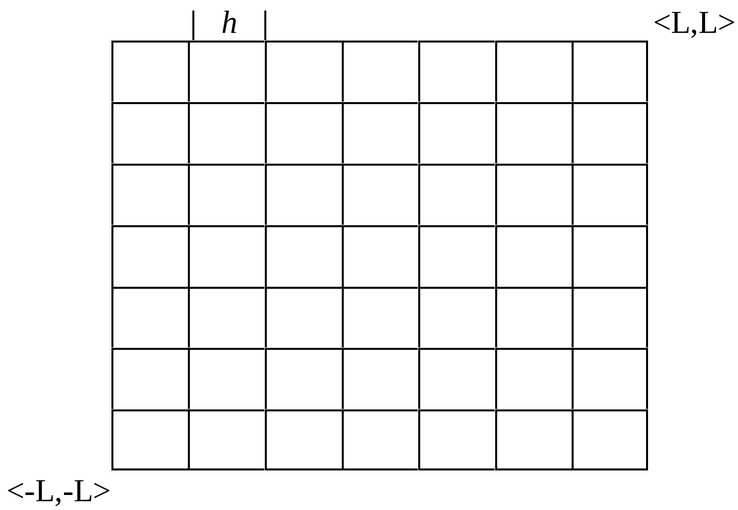
4. False. Let $A = [0,1]$. Let $B = [0,1] \cup [2,3]$. Let $C = \mathbf{R}^1$.

6. Wolog we may suppose that $B \setminus A$ is non-empty, i.e., B is strictly bigger than A. Suppose to the contrary that B were disconnected. Then, there would exist a non-trivial disjoint open decomposition of B, say sets C and D. Since A is connected, we must have either $A \subset C$ or else $A \subset D$. (Otherwise the sets $A \cap C$ and $A \cap D$ would form non-trivial disjoint open decomposition of A.) Wolog $A \subset C$. Therefore $(B \setminus A) \cap D$ must be non-empty.

But if $x \in B \setminus A$, then x must be a limit point of A. But, as a consequence of theorem 5.5D if x is a limit point of A and if x lies in an open set D, then $D \cap A \neq \varnothing$. This contradicts the fact that $A \subset C$ and $C \cap D = \varnothing$.

Section 6.3

1. Suppose that B is a bounded subset of $\mathbf{R}^2$. Then there exists L such that B is contained inside the square S with corners (diametrically opposite) $\langle -L,-L \rangle$ and $\langle L,L \rangle$. Let $\varepsilon > 0$. Let $h = \frac{\varepsilon}{\sqrt{2}}$. Cover S with a rectangular grid of (closed) cells whose parallel sides are



distance h apart. Let $N = [[\frac{2L}{h}]]+1$. Then, there are at most N^2 cells in the grid, each cell has diameter ε, and the collection of cells in the grid covers S which contains B.

3. Example 1. $S = \{s_n\}$ where each $s_n = \langle 1/n, 0, 0, 0, 0, \ldots \rangle$, for n = 1, 2, 3, . . . Then, each $s_n \in \ell^2$ and $\| s_n - s_m \|_2 = |1/n - 1/m|$. It is easily seen that each subsequence of S is Cauchy.

   Example 2. $S = \{s_n\}$ where each $s_n = \langle 0, 0, 0, \ldots, 1/n, 0, 0, 0, 0, \ldots \rangle$, where the single non-zero entry is in the n^{th} slot, for n = 1, 2, 3, . . . Then, each $s_n \in \ell^2$ and $\| s_n - s_m \|_2 = (1/n^2 - 1/m^2)^{1/2}$. It is easily seen that each subsequence of S is Cauchy.

4. Let $S = \{x_1, x_2, \ldots, x_n\}$ be a finite subset of M. Let $\varepsilon > 0$. Choose sets $A_k = B(x_k, \varepsilon/2)$ for k = 1 .. n. Then, the sets $\{A_1, A_2, \ldots, A_n\}$ cover S and each A_k has diameter ε.

5. Let $\varepsilon > 0$. Since M is total bounded there exist sets $\{A_1, A_2, \ldots, A_n\}$ in M such that $\mathrm{diam}(A_k) < \varepsilon$ and $M \subset \bigcup_{k=1}^{n} A_k$. But since $A \subset M$, we also have for $B_k = A_k \cap A$ that

diam(B_k) < ε and $A \subset \bigcup_{k=1}^{n} B_k$.

7. Denote A = A_0. We will show that there exists a sequence in A_0 which is Cauchy. Since A_0 is bounded there exists a closed bounded interval J_0 such that $A_0 \subset J_0$. Let diam(J_0) = d.

Find the midpoint of J_0 and divide J_0 into two subintervals of equal width. Since A_0 is infinite, at least one of the these subintervals must contain an infinite number of points. Choose one of them and call it J_1. Note that diam(J_1) = d/2. Let $A_1 = J_1 \cap A_0$.

Find the midpoint of J_1 and divide J_1 into two subintervals of equal width. Since A_1 is infinite, at least one of the these subintervals must contain an infinite number of points. Choose one of them and call it J_2. Note that diam(J_2) = d/4. Let $A_2 = J_2 \cap A_1$.

Proceed inductively to find for each k = 1, 2, 3, ... sets J_k such that each $J_k \subset J_{k-1}$ and diam(J_k) = d/2_k and $A_k = J_k \cap A_{k-1}$ where each A_k is infinite. The sets {J_k } satisfy the nested interval theorem and hence the property $\bigcap_{k=1}^{\infty} J_k$ is a singleton.

Since each $J_k \cap A_0 \neq \varnothing$ and diam(J_k) = d/2^k and $\bigcap_{k=1}^{\infty} J_k \in J_k$, then we see that $\bigcap_{k=1}^{\infty} J_k$ is a cluster point of A_0.