Math 5319 Exam I February 12, 2001

Answer the problems on separate paper. You do not need to rewrite the problem statements on your answer sheets. Do your own work. Show all relevant steps which lead to your solutions. Retain this question sheet for your records.

In-Class

1. For each of the following sets (S, G, X, Q) determine whether the sets are:

(a) open;	(b) closed;	(c) connected;
(d) bounded;	(e) totally bounded;	(f) compact.

Also, find the diameter of each set.

i. $S = \{ \langle x,y\rangle \in \boldsymbol{R}^2 : 0 < xy < 1 \}$ $(S \subset \boldsymbol{R}^2)$

ii. $G = \bigcap_{n=1}^{\infty} G_n, \quad G_n = (0, 1 + \frac{1}{n})$ $(G \subset \boldsymbol{R}^1)$

iii. $X = Y \cup \{2\} \cup Z$, where $Y = \{ \sum_{k=0}^{n} \frac{1}{2^k} \}_{n=0}^{\infty}$, and $Z = \{ \frac{2n+1}{n} \}_{n=1}^{\infty}$

$(X \subset \boldsymbol{R}^1)$

iv. $Q = \bigcup_{n=1}^{\infty} Q_{2^n}$, where, for k even, $Q_k = \{ \frac{1}{k}, \frac{3}{k}, \frac{5}{k}, \cdots, \frac{k-1}{k} \}$

$(Q \subset \boldsymbol{R}^1)$

2. Let $\langle M, \rho\rangle$ be a metric space. Let A and B be subsets of M. Prove that if A and B are are compact, then $A \cup B$ is compact.

3. Let J be an interval in $\boldsymbol{R}^1$. Let f and g be real-valued, uniformly continuous functions on J. Prove that $f + g$ is uniformly continuous on J.

4. Let $\langle M, \rho\rangle$ be a metric space. Let f be a continuous, real-valued function on M. Prove that if M is connected, then the image, $f(M)$, is an interval.

5. Let f be a bounded real-valued function on a closed bounded interval $[a, b]$. Prove or disprove each of the following propositions:

(a) If $f \in \Re[a, b]$, then $f^2 \in \Re[a, b]$.
(b) If $f^2 \in \Re[a, b]$, then $f \in \Re[a, b]$.

Take-Home (Due Friday, 5:00 pm)

Do five (5) of the following problems:

1. Let $E \subset \boldsymbol{R}^1$. Show that if E is uncountable, then there exists a point $x_0 \in E$ such that x_0 is a limit point of E.

2. Let $\langle M, \rho \rangle$ be a metric space and let $x_0 \in M$. Define $f : M \to \boldsymbol{R}^1$ by $f(x) = \rho(x, x_0)$. Show that f is uniformly continuous on M.

3. Let f be a continuous, one-to-one map from $\boldsymbol{R}^1$ into $\boldsymbol{R}^1$. Show that f is a homeomorphism.

4. Let f be a bounded real-valued function on a closed bounded interval $[a, b]$. Let $f \in \mathfrak{R}[a, b]$. Let $M = \max\limits_{x \in [a,b]} f(x)$. Show only using material from 7.1-7.3, that $\int_a^b f \le \int_a^b M$.

5. Let $f(x) = 2 + x$ on $[0,2]$. Let σ_n be the uniform partition of $[0,2]$ into subintervals of equal width. Find $U[f; \sigma_n]$. Find $\int_0^2 f$ by finding $\lim\limits_{n \to \infty} U[f; \sigma_n]$.

6. Let $Q^* = \bigcup\limits_{n=1}^{3} Q_{2^n}$, where, for k even, $Q_k = \{\frac{1}{k}, \frac{3}{k}, \frac{5}{k}, \cdots, \frac{k-1}{k}\}$. Let $p(x) = \frac{3456}{35}x^3 - \frac{4272}{35}x^2 + \frac{1412}{35}x$. Define f on $[0,1]$ by

$$f(x) = \begin{cases} 2^n & x \in Q_{2^n} \subset Q^* \\ p(x) & x \notin Q^* \end{cases}$$

. Find $\omega[f; x]$ for each $x \in [0,1]$.