Theorems from Algebra

1. Consequences (Field Axioms)
   a. **Theorem**: $(a, b, c \in \mathbb{R}$ AND $a = b) \Rightarrow (a + c = b + c)$
   b. **Theorem**: $(a, b, c \in \mathbb{R}$ AND $a = b) \Rightarrow (ac = bc)$
   c. **Theorem**: $(a, b, c \in \mathbb{R}$ AND $a + c = b + c) \Rightarrow (a = b)$
   d. **Theorem**: $(a, b, c \in \mathbb{R}$ AND $ac = bc$ AND $c \neq 0) \Rightarrow (a = b)$
   e. **Theorem**: $(a, b \in \mathbb{R}$ AND $a + b = 0) \Rightarrow (a = -b$ AND $b = -a)$
   f. **Theorem**: $(a \in \mathbb{R}) \Rightarrow a \bullet 0 = 0$
   g. **Theorem**: $(a, b \in \mathbb{R}$ AND $a \bullet b = 0) \Rightarrow (a = 0$ OR $b = 0)$
   h. **Theorem**: $(a, b \in \mathbb{R}) \Rightarrow$
      i. $a = -(-a)$
      ii. $-(a + b) = -a + -b$
      iii. $(-a)b = -(ab)$
      iv. $(-a)(-b) = ab$
   i. **Theorem**: $(a, b, c, d \in \mathbb{R}$ AND $b, d \neq 0) \Rightarrow (a/b = c/d \Leftrightarrow ad = bc)$
   j. **Theorem**: $(a, b, c \in \mathbb{R}$ AND $b, c \neq 0) \Rightarrow (ac / bc = a/b)$
   k. **Theorem**: $(a, b \in \mathbb{R}$ AND $a, b \neq 0$ AND $ab = 1) \Rightarrow (a = b^{-1}$ AND $b = a^{-1})$
   l. **Theorem**: $(a, b \in \mathbb{R}$ AND $a, b \neq 0) \Rightarrow$
      i. $a/a = 1$
      ii. $a/1 = a$
      iii. $1/a = a^{-1}$
      iv. $1 / (1/a) = a$
      v. $(-a)/b = - a/b = a/(-b)$
      vi. $(-a)/(-b) = a/b$
   m. **Theorem**:$(a, b, c, d \in \mathbb{R}$ AND a, b, c, d ... $0) \Rightarrow$
      i. $(1/a) \bullet (1/b) = 1/ (ab)$
      ii. $(a/b) \bullet (c/d) = (ac) / (bd)$
      iii. $a/c + b/c = (a+b) /c$
      iv. $a/b + c/d = (ad + bc) / (bd)$
      v. $a/b - c/d = (ad-bc) / (bd)$
      vi. $1 / (a/b) = b/a$
      vii. $(a/b) / (c/d) = (a/b) \bullet (d/c)$
2. Consequences (Order Axioms)
   a. **Lemma**: $(a \in \mathbb{R}) \Rightarrow$
      i. $(a > 0) \Leftrightarrow (a \in \mathbb{R}^{+})$
      ii. $(a < 0) \Leftrightarrow (a \in \mathbb{R}^{-})$
   b. **Lemma**: $(a, b \in \mathbb{R}) \Rightarrow$
      i. $a \leq a$
      ii. $(a \leq b$ AND $b \leq a) \Leftrightarrow (a = b)$
   c. **Theorem**: $(a, b \in \mathbb{R}) \Rightarrow$
      i. $(a > 0$ AND $b < 0) \Rightarrow (ab < 0)$
      ii. $(a < 0$ AND $b < 0) \Rightarrow (ab > 0)$
      iii. $a^2 \geq 0$
   d. **Theorem**: $(a, b, c \in \mathbb{R}) \Rightarrow$
      i. $(a < b$ AND $b < c) \Rightarrow (a < c)$
      ii. $(a < b) \Leftrightarrow (a + c < b + c)$
      iii. $(a < b$ AND $c > 0) \Leftrightarrow (ac < bc$ AND $c > 0)$
      iv. $(a < b$ AND $c < 0) \Leftrightarrow (ac > bc$ AND $c < 0)$
3. Consequences
   a. Theorem: $(a, b \in \mathbb{R}) \Rightarrow$
      i. $(|a| < b) \Leftrightarrow (-b < a < b)$

ii. $(|a| > b) \Leftrightarrow (a > b \text{ OR } a < -b)$