

Answer the problems on separate paper. You do not need to rewrite the problem statements on your answer sheets. Do your own work. Show all relevant steps which lead to your solutions. Retain this question sheet for your records.

**Part I** Do 4 (four) problems. Each problem is worth 10 points.

I-1. Let  $\text{Im } z < 0$ . Prove that  $\text{Im } 1/z > 0$ .

I-2. Prove  $||z_1| - |z_2|| \leq |z_1 - z_2|$ .

I-3. Write the number  $\frac{2-2i}{\sqrt{3}-i}$  in rectangular  $(a + bi)$  and polar  $(re^{i\theta})$  form.

I-4. Sketch the graph of  $z(t) = e^{(1-i)t}$ ,  $0 \leq t \leq 1$ .

I-5. Find all values of  $(1+i)^{\frac{1}{3}}$ .

I-6. Consider the sets:

A)  $\{z : |z+i| < 2\} \cup \{z : |z-i| < 2\}$   
 B)  $\{z : \text{Re } z > 1\} \cup \{z : \text{Re } z < -1\}$   
 C)  $\{z : |z-2| < 4\} \setminus \{z : |z-2| < 1\}$

Which sets are: i) open, ii) connected, iii) bounded, iv) convex?

**Part II** Do 2 (two) problems. In this section ( $z = x + iy$ ). Each problem is worth 10 points.

II-1. Determine where the following functions are analytic:

i)  $f(z) = \frac{z}{1-z}$       ii)  $g(z) = z + 2\bar{z}$       iii)  $h(z) = x^2 + y + i(2xy - x)$

II-2. Show that  $f(z) = 3x^2 + 2x - 3y^2 - 1 + i(6xy + 2y)$  is analytic on  $\mathbb{C}$  and find  $f'(z)$ .

II-3. Let  $D$  be a domain and let  $f \in A(D)$ . Let  $f(z) = u(x,y) + i v(x,y)$  on  $D$ . Show that if  $u(x,y) = v(x,y)$  on  $D$ , then  $f$  is constant.

II-4. Show that  $u(x,y) = e^{-x} \sin y$  is harmonic on  $\mathbb{C}$ . Find the harmonic conjugate.

**Part III** Do 1 (one) problem. Each problem is worth 10 points.

III-1. Find all solutions of  $e^z = \sqrt{3} + i$ .

III-2. Find all values of  $(1 + i)^{1-i}$ .

**Part IV** Do all 3 (three) problems. Each problem is worth 10 points.

IV-1. Evaluate the following integrals:

i)  $\int_{\Gamma} \bar{z} dz$ , where  $\Gamma$  is the part of the parabola  $y = x^2$  from the point  $(0,0)$  to  $(1,1)$ .

ii)  $\int_{\Gamma} (z + z^3) dz$ , where  $\Gamma$  is the part of the curve  $y = \sin 4\pi x$  from the point  $(0,0)$  to  $(1,0)$ .

IV-2. Let  $\Gamma$  be the circle  $\{z : |z| = 2\}$  traversed once in the positive direction. Evaluate the integral  $\int_{\Gamma} f(z) dz$ ,

where:

i)  $f(z) = \frac{\cos z}{z^3 + 9z}$                       ii)  $f(z) = \frac{\sin z}{z^2(z-4)}$

IV-3. Consider the square  $S$  with vertices  $A(4+4i)$ ,  $B(-4+4i)$ ,  $C(-4-4i)$  and  $D(4-4i)$ . Let  $f \in A(S)$ . Suppose that for (all)  $z \in \partial S$  we have the following bound  $|f(z)| \leq 24/7$ . Determine whether or not we can have  $f''(0) = 1/2$ .

**Part V** Do 2 (two) problems. Each problem is worth 10 points.

V-1. Find the first five terms of the Taylor's series centered at  $z = 0$  for the following functions:

i)  $f(z) = \frac{\sin z}{1-z^2}$                       ii)  $f(z) = \frac{z}{(1-z)^2}$

V-2. Find the Laurent series on the given annulus for the following functions:

i)  $f(z) = \frac{1}{z+z^2}$  on  $0 < |z| < 1$                       ii)  $f(z) = \frac{e^{z^2}-1}{z^3}$  on  $|z| > 0$ .

V-3. Find and classify the finite isolated singularities for the following functions. Also, compute the residues at each of those singularities for those functions.

i)  $f(z) = \frac{z^2}{e^z - 1}$                       ii)  $f(z) = \frac{z^2 - 2z + 1}{z^4 - 1}$                       iii)  $f(z) = \frac{e^z - 1}{z^3}$

**Part VI** Do 4 (four) problems. Each problem is worth 10 points.

VI-1. Evaluate the following integrals:

i)  $\int_{\Gamma} \frac{e^z}{z(z-2)} dz$ , where  $\Gamma$  is the circle  $\{z : |z| = 3\}$  traversed once in the positive direction.

ii)  $\int_{\Gamma} \frac{1}{z^2 \sin z} dz$ , where  $\Gamma$  is the circle  $\{z : |z| = 1\}$  traversed once in the positive direction.

VI-2. Evaluate, using residue arguments, the following integral:

$$\int_0^{2p} \frac{dq}{2 + \cos q}.$$

VI-3. Evaluate, using residue arguments, the following integral:

$$\int_0^{\infty} \frac{x^2 + 1}{x^4 + 1} dx.$$

VI-4. Evaluate, using residue arguments, the following integral:

$$\int_0^{\infty} \frac{\cos 3x}{x^2 + 1} dx.$$

VI-5. Evaluate, using residue arguments, the following integral:

$$\int_0^{\infty} \frac{\cos x - 1}{x^2} dx.$$

VI-6. Evaluate, using residue arguments, the following integral:

$$\int_0^{\infty} \frac{\sqrt{x}}{x^2 + 1} dx.$$

**Part VII** Do 4 (four) problems. Each problem is worth 12 points.

VII-1. Without using a calculator, determine how many of the roots of the polynomials  $p(z) = z^7 - 3z^3 + 9z^2 + 3$  lie in the annulus  $\{z : 1 < |z| < 2\}$ .

VII-2. Describe the image of the rectangle  $\{z : 1 < \operatorname{Re} z < 2, 0 < \operatorname{Im} z < p\}$  under the mapping  $w = e^z$ .

VII-3. Describe the image of the first quadrant ( $Q_I$ ) under the mapping  $w = \frac{z-i}{z}$ .

VII-4. Describe the image of the circle  $\{z : |z - 2i| = 2\}$  under the mapping:

i)  $w = z + 3i$     ii)  $w = \frac{1}{z-4i}$     iii)  $w = 5z$

VII-5. Find (construct a formula for) the Möbius transformation which maps:

$$0 \rightarrow -1$$

$$1 \rightarrow \infty$$

$$\infty \rightarrow 1$$