MATH 4356-001 Exam III November 30, 1999

Answer the problems on separate paper. You do not need to rewrite the problem statements on your answer sheets. Do your own work. Show all relevant steps which lead to your solutions. Retain this question sheet for your records.

Part I. Do 5 (five) of the following problems.

1. [12 pts] Find the sum of the series $\sum_{k=2}^{\infty}\left(\frac{1-i}{2}\right)^k$.

2. [12 pts] Determine whether the series $\sum_{k=1}^{\infty} k^2\left(\frac{i}{1-i}\right)^k$ converges.

3. [12 pts] Find the Taylor series centered at i for $\frac{1}{1-z}$ and determine its (the Taylor series) radius of convergence.

4. [12 pts] Find the radius of convergence of the power series $\sum_{k=1}^{\infty}\frac{(2+i)^k(z-1)^k}{k^2}$.

5. [12 pts] Find and classify each of the finite isolated singularities for each of the following functions (classify as either removable, pole [and order], or essential):

(a) $\frac{z^3-1}{z^3+1}$ b) $\frac{1}{\sin z}$ c) $(\frac{1}{z}+1)\sin\frac{1}{z}$

6. [12 pts] For each of the functions in Problem 5., if the singularity at the point at infinity is isolated, classify the singularity at the point of infinity.

7. [12 pts] Let f and g be analytic on a neighborhood of 0. Suppose that f has a zero at 0 of order $m \geq 1$ and that g has a zero at 0 of order $n \geq 1$. Determine whether each of the following statements is true or false. Supply a counterexample for each statement that is false.

(a) $f + g$ has a zero at 0 of order $m + n$
(b) $f + g$ has a zero at 0 of order $\min(m,n)$
(c) $f\,g$ has a zero at 0 of order $m + n$
(d) $f\,g$ has a zero at 0 of order mn
(e) $f \circ g$ has a zero at 0 of order mn
(f) $f \circ g$ has a zero at 0 of order $\max(m,n)$

Part II.

1. [12 pts] Find the residue at each of the isolated singularities for each of the following functions:

(a) $\dfrac{z-1}{z^2+2z+2}$ (b) $\dfrac{1-\cos z}{z^3}$ (c) $z^2 e^{-\frac{2}{z}}$

Part III. Do 1 (one) of the following problems.

1. [12 pts] Find the integral . $\displaystyle\int_{|z|=2} \frac{\sin 2z}{1-\cos z}\,dz$

2. [12 pts] Find the integral. $\displaystyle\int_{|z-2|=2} \frac{z-2i}{z^2-2i}\,dz$ [Note: $z^2 - 2i = z^2 - (1+i)^2$]

Part IV. Do 2 (two) of the following problems.

Use residues to evaluate the following integrals. If you introduce a contour integral, clearly describe the contour. [If the contour is composed of subcontours, clearly describe each of the subcontours.] If your argument introduces a limit process for the contour integral, clearly detail the limit behavior of your integral over the contour [or subcontours, as the case maybe].

1. [12 pts] $\displaystyle\int_0^{2p} \frac{dq}{5+4\sin q}$

2. [12 pts] $\displaystyle\int_0^{\infty} \frac{dx}{(x^2+1)^2}$

3. [12 pts] $\displaystyle\int_0^{\infty} \frac{x\sin 2x}{x^2+3}\,dx$