MATH 4356-001 Exam II October 28, 1999

Answer the problems on separate paper. You do not need to rewrite the problem statements on your answer sheets. Do your own work. Show all relevant steps which lead to your solutions. Retain this question sheet for your records.

1. [10 pts] Let $R$ be the rectangle with vertices $0$, $2$, $2+\frac{1}{2}\pi i$, $\frac{1}{2}\pi i$. Sketch the image of $R$ under the exponential map $e^z$.

2. [10 pts] Find all solutions of $e^{iz} = 2$.

3. [10 pts] Determine the domain of analyticity of the function $\mathrm{Log}\,(1 - 2iz)$.

4. [10 pts] Recall that given a domain $D$ and a continuous, non-vanishing function $f$ on $D$, we say that a continuous function $g$ on $D$ is a *branch-of-log* of $f$ on $D$ if $e^g = f$ on $D$.

   Prove that if $g$ and $h$ are both branches-of-log of $f$ on $D$, then there exists an integer $k$ such that $h = g + 2\pi k i$ on $D$.

5. [10 pts] Find all values of $2^{3i}$.

6. [12 pts] Let $\Gamma$ be a smooth contour with parametrization $z(t) = t + i t^2$, $0 \le t \le 1$. Find

   a) $\displaystyle\int_\Gamma \bar{z}\, dz$ b) $\displaystyle\int_\Gamma z^3\, dz$

7. [20 pts] Let $\Gamma$ be the circle $|z - 2| = 2$ traversed once in the positive sense. Find

   a) $\displaystyle\oint_\Gamma \frac{1}{z^2+1}\, dz$ b) $\displaystyle\oint_\Gamma \frac{1}{z^2-1}\, dz$ c) $\displaystyle\oint_\Gamma \frac{1}{(z^2+1)^2}\, dz$ d) $\displaystyle\oint_\Gamma \frac{1}{(z^2-1)^2}\, dz$

8. [10 pts] Suppose that $f$ is an entire function such that $\mathrm{Im}\, f(z) > 0$ for all $z$. Show that $f$ must be constant. Hint: Consider $g = e^{if}$.

9. [10 pts] Let $D = \{ z : |z| < 1\}$. Let $f$, $g$ be continuous, non-vanishing functions on $D \cup \partial D$ which are analytic on $D$. Suppose that for $z \in \partial D$ that $|f(z) / g(z)| = 1$. Prove that there exists a constant $k$ such that $f = k g$.