1. (12 pts) Complete each of the following statements:

   a. Let $\alpha : S \rightarrow T$. $\alpha$ is onto if and only if . . . .
   
   b. Let $\alpha : S \rightarrow T$. $\alpha$ is invertible if and only if . . . .
   
   c. Let $S$ be a non-empty set and $*$ an operation on $S$. $e \in S$ is an identity element for $*$ if and only if . . . .
   
   d. Let $(G,*)$ be a group. A subset $H$ of $G$ is a subgroup of $G$ if and only if . . . .

2. (12 pts) Consider the following sets $S$ and equations, each of which defines a rule $*$ for $S$. In each case, determine whether $*$ defines an operation on $S$. If $*$ does not define an operation on $S$, state why not.

   a. $S = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \right\} \mid a, b, c \in \mathbb{Z} \}, \, *$ is matrix multiplication
   
   b. $S = \left\{ \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} \right\} \mid a, b, c \in \mathbb{Z} \}, \, *$ is matrix multiplication

3. (12 pts) Consider the following equation, which defines an operation $*$ on $\mathbb{Z}$:

   $m * n = m + n + mn$

   a. Determine if $*$ is associative. If not, state why not.
   
   b. Determine if $*$ has an identity. If not, state why not.
   
   c. Determine if $*$ is commutative. If not, state why not.

4. (12 pts) Complete the following Cayley table so that $G = \{a, b, c, d\}$ with $*$ is a commutative group:

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>d</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>b</td>
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<tr>
<td>c</td>
<td>a</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>d</td>
<td></td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>
5. (18 pts) Consider the following sets $G$ with associative operations $*$. In each case, determine whether $(G, *)$ is a group. If $(G,*)$ is not a group, state why it is not.

a. $G = \{0, 1, 2, 3, 4, \cdots\}$, $*$ is addition of integers
b. $G = \mathbb{Q} \setminus \{0\}$, i.e., $G$ is the set of non-zero rational numbers, $*$ is multiplication of rational numbers
c. $G = M(\mathbb{Z})$, i.e., $G$ is the set of mappings from $\mathbb{Z}$ to $\mathbb{Z}$, $*$ is addition of functions

6. (12 pts) Write each of the following as a single cycle or as a product of disjoint cycles:

a. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 6 & 1 & 2 & 4 \end{pmatrix}$

b. $(1 \ 3 \ 2 \ 6) (2 \ 5 \ 1) (3 \ 1 \ 7 \ 2)$

c. $(2 \ 5 \ 4 \ 1) \cdot (1 \ 3 \ 5 \ 2) (1 \ 4 \ 3)$

7. (12 pts) It is known that $\mathbb{R}^2$ with component-wise addition (i.e., $(a, b) + (c, d) = (a + c, b + d)$) forms a group. Show that the subset $H = \{(x,y) \mid x=y, x,y \in \mathbb{R}\}$ is a subgroup of $\mathbb{R}^2$.

8. (12 pts) Identify the symmetry group of the figure to the right, i.e., construct a list (or a table) which identifies each of the elements which belong to the symmetry group of the figure. The figure consists of a square centered at the origin minus 4 slits of equal length on the diagonals and 2 slits of equal length on the $x$-axis.

Also, identify which elements in the symmetry group of the figure are self-inversive, i.e., which elements in the symmetry group of the figure are their own inverses.