MATH 3360-001 Exam III July 1, 2005

Answer the problems on separate paper. You do not need to rewrite the problem statements on your answer sheets. Do your own work. Show all relevant steps which lead to your solutions. Retain this question sheet for your records.

1. a. (2 pts) Let G, H be groups and let $\theta : G \to H$ be a homomorphism. Complete the following statement: the kernel of θ is . . . .

   b. (2 pts) Complete the following statement: an integral domain D is said to be an *ordered integral domain* if . . . .

2. (6 pts) Let $\mathbb{R}^2$ denote the set of ordered pairs of real numbers, where $\mathbb{R}^2$ is considered as a group with the usual component-wise addition. Let $a = \begin{bmatrix} -1 & 0 \\ 2 & 0 \end{bmatrix}$ and let $\lambda : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $\lambda x = xa$. Prove that λ is a group homomorphism. Find $\ker \lambda$.

3. (6 pts) Let $P_{\mathbb{Z}}[x]$ denote the set of polynomials with integer coefficients, where $P_{\mathbb{Z}}[x]$ is considered as a group with the usual polynomial addition. Let $\lambda : P_{\mathbb{Z}}[x] \to P_{\mathbb{Z}}[x]$ be given by $\lambda(f) = f'$. Prove that λ is a group homomorphism. Find $\ker \lambda$.

4. (6 pts) Construct the Cayley table for the quotient group $\mathbb{Z}_{16} / <[4]>$.

5. (6 pts) In the ring $M(2, \mathbb{Z}_{12})$ compute $\left(\begin{bmatrix} 7 & 3 \\ 2 & 10 \end{bmatrix} \oplus \begin{bmatrix} 8 & 4 \\ 2 & 6 \end{bmatrix} \right) \odot \begin{bmatrix} 9 & 3 \\ 4 & 11 \end{bmatrix}$.

6. (6 pts) Prove that if R is a ring and if $a, b, c \in R$, then $a + b = a + c$ implies $b = c$.

7. (6 pts) Prove that if R is a integral domain and if $a, b, c \in R$ and $a \neq 0$, then $ab = ac$ implies $b = c$.

8. (6 pts) Identify all of the zero divisors of the ring $\mathbb{Z}_4 \times \mathbb{Z}_8$.

9. Let R be the ring $P_{\mathbb{Z}}[x]$ and consider the subset $S = P_E[x]$ of R consisting of polynomials with even coefficients and the subset $T = P_O[x]$ consisting of polynomials with odd coefficients.

a. (6 pts) Prove that S is a subring of R.

b. (3 pts) Prove that T is not a subring of R.

10. (6 pts) Prove that if R is a ring and if $a, b \in R$, then $-(ab) = (-a)b$.

11. (6 pts) Prove that if R and S are isomorphic rings and if R has no zero-divisors, then S has no zero divisors.

12. (4 pts) For each part provide an example of a ring which satisfies the condition or state that one does not exist. Note in part d. the subscript p denotes a prime. (No explanation is required for stating the example or the non-existence of such an example.)

a. A non-commutative integral domain
b. A commutative ring without a unity
c. A integral domain which is not a field other than $\mathbb{Z}$
d. A finite field other than $\mathbb{Z}_p$

13. (5 pts) List 5 ring properties which every ring isomorphic to $\mathbb{Z}$ (ring isomorphism) must possess, which are not necessarily valid for every ring.

14. Write in the form $a + bi$, $a, b \in \mathbb{R}$

a. (4 pts) $\dfrac{2-i}{3+2i}$ b. (4 pts) $(-2+2i)^8$

15. Let U_6 denote the set of all complex 6th roots of unity.

a. (4 pts) Determine U_6, i.e., identify all of the elements of U_6.

b. (4 pts) Represent the elements of U_6 graphically, i.e., plot them in $\mathbb{C}$.

c. (4 pts) For each element of U_6 identify its multiplicative inverse.

d. (4 pts) Let V_6 denote the (unique) subgroup of U_6 of order 3. Determine V_6.