Answer the problems on separate paper. You do <u>not</u> need to rewrite the problem statements on your answer sheets. Do your own work. Show all relevant steps which lead to your solutions. For no question (except question 12) is True/Yes or False/No by itself sufficient as the answer. <u>Retain</u> this question sheet for your records.

1. (6 pts)   a.   Let R with operations $+$, $\bullet$ and S with operations $+$, $\bullet$ be rings. Complete the following statement: R is isomorphic to S if and only if . . . .

             b.   Let R with operations $+$, $\bullet$ be a ring. Complete the following statement: R is an integral domain if and only if . . . .

2. (6 pts)   Let $s = \begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix}$ and let $\lambda : M(2, Z) \rightarrow M(2,Z)$ be given by $\lambda(x) = sx$.

             Prove that λ is a group homomorphism. Find ker λ.

3. (6 pts)   Determine the order of the quotient group G/N where $G = Z_{24} \times Z_6$ and N is the subgroup of G given by:

             a. $N = <[6]> \times <[3]>$ b. $N = <([6], [3])>$

4. (6pts)   Let G be an Abelian group and let N be a normal subgroup of G. Prove that the quotient group G/N is Abelian.

5. (8 pts)   Find a representative for each possible (group) homomorphic image of G for

             a. $G = Z_{10}$ b. $G = Z_{16}$

6. (6 pts)   Prove: $Z_{24} / <[3]> \approx Z_3$.

7. (6 pts)   In $M(2, Z_{12})$, compute:

$$\left(\begin{bmatrix} [3] & [9] \\ [7] & [6] \end{bmatrix} \oplus \begin{bmatrix} [4] & [0] \\ [3] & [8] \end{bmatrix} \right) \odot \begin{bmatrix} [10] & [5] \\ [2] & [11] \end{bmatrix}$$

8. (6 pts)   Let R be isomorphic to S. Prove that if R is an integral domain, then S is also an integral domain.

9. (8 pts)   Identify all of the zero divisors of $M(2,Z_6)$.

10. (8 pts)   Let $S = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \middle| a, b \in Z \right\}$.

        a.   Determine whether S is a subring of M(2,Z).

        b.   If the answer to a. is true, then determine whether S is an integral domain. If the answer to a. is false, then skip this part.

11. (8 pts)   Let E be the set of even integers. Define addition for this set E to be the usual addition of integers. Define multiplication for this set E by $m \cdot n = \dfrac{mn}{2}$

        a.   Determine whether E with the above addition and multiplication is a ring.

        b.   If the answer to a. is true, then determine whether E with the above addition and multiplication is an integral domain. If the answer to a. is false, then skip this part.

12. (8 pts)   For each part provide an example of a ring which satisfies the condition or state that one does not exist. (No explanation is required for stating the example or the non-existence of such an example.)

        a.   A non-commutative ring
        b.   A integral domain which is ordered but not well-ordered
        c.   A finite integral domain which is ordered
        d.   A well-ordered field

13. (8 pts)   Let E and F be fields and let $\theta : E \rightarrow F$ be an isomorphism. If $a \in$ E and $a \neq$ 0, then prove that $\theta(a^{-1}) = \theta(a)^{-1}$.

14. (8 pts)   Suppose that D is an ordered integral domain. Prove that if $a > b > 0$, then $a^2 > b^2$.

15. (6 pts)   List six ring properties that hold for each ring isomorphic to Q, but not for every ring.