Guidelines for Essays for Math 3360

General

Essays should be prepared on a word processor. Their length should be approximately 150 words (50 - 200 words), unless otherwise explicitly stated. Each essay should be labeled with title and author at the beginning of the essay. There does <u>not</u> need to be a separate cover page.

Content

My primary concern in reading your essays will be with their content. Does your essay clearly, accurately and succinctly describe the objective of the assignment? Does your essay appropriately treat any subtleties which may lie within the scope of the objective?

Style

If writing mathematics is new for you, there are some guidelines on style which you may find helpful. Specifically, there is an article *Manual for Authors of Mathematical Papers* [1] which makes suggestions about style in mathematics. The suggestions about style, notation, citations, references and bibliography are relevant to the essays you will be writing for this course. The article is somewhat dated in its discussion of typesetting.

Form

A secondary concern in reading your essays will be with form. Specifically, I expect that spelling and punctuation are accurate and proper and I expect appropriate grammar.

Bibliography

1. J. L. Dobb, L. Carlitz, F. A. Ficken, G. Piranian and N. E. Steenrod, *Manual for Authors of Mathematical Papers*, Bull. Amer. Math. Soc. **68** (1962), 429-444.

Writing Assignment #1 Due January 31

Consider the following definition:

Definition: A set $O \subseteq \mathbb{R}^2$ is *open* if for each point $p \in O$ there exists a positive radius r such that the open ball $B(p,r) = \{ q : |q - p| < r \} \subseteq O$.

Write a proof for the following theorem:

Theorem. Let $O_1, O_2, \ldots O_n$ be open subsets of R^2. Let O be the intersection of the sets $O_1, O_2, \ldots O_n$. Then, O is also open subset of R^2.

Also, add a remark or a note which explains why the theorem is not true for an infinite number of open subsets of R^2.

Writing Assignment # 2 Due February 14

Let $P(n)$ be a statement which depends on n, where n is a natural number. Suppose we have the following theorem:

Theorem. $P(n)$ is true for all n, $n = 1,2,3,\ldots.$

Describe the steps which would be required to prove the theorem using mathematical induction.

Use mathematical induction to prove the following theorem:

Theorem. Let $\{r_k\}$, $k = 1,2,3,\ldots$ be a sequence of positive real numbers. For each $n = 1,2,3,\ldots$ let $m_n = \min\limits_{1 \le k \le n} \{r_k\}$. Then, for each $n = 1,2,3,\ldots$, we have $m_n > 0$.

(Note, this was the key step which was needed in the proof of the theorem about the intersections of finite collections of open sets.)

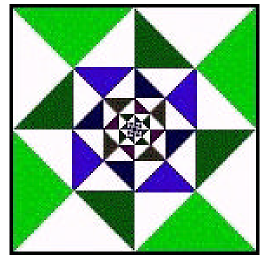
Writing Assignment #3 Due February 28

In the text, on page 51 Durban states:

> Consider a square, a rectangle and a parallelogram (Figure 8.3). Any motion [which leaves the figure invariant] of one of the figures will permute the vertices of the figure among themselves and the sides among themselves. Moreover, any motion will be completely determined by the way it permutes the vertices.

Explain why the vertices of such a figure must be permuted among themselves by a motion which leaves the figure invariant. Clearly, the fact that the figure is left invariant under the motion means that the figure must be mapped by the motion back to itself. But why couldn't the vertices be mapped to non-vertical points? For instance, why couldn't every point on the square be mapped to a new position on the square which was "45E" further along on the square. In this case the vertices would end up at the mid-points of the sides and the mid-points of the sides would land on the vertices.

Identify the symmetry group of the "four-sided" figure:



Writing Assignment #4 Due March 22

Each of the familiar number systems **Z**, **Q**, **R** and **C** with operations + and · are integral domains. That means that the following property holds: A commutative ring *S* with unity is an *integral domain* if whenever *a*, *b* 0 *S*, then $a \cdot b = 0$ implies that either $a = 0$ or $b = 0$. (Note a commutative ring *S* is a set *S* with two operations + and · such that $(S,+)$ is an abelian group and such that the operation · is associative, commutative and distributive over +.) As we shall see the fact that **Z** is an integral domain comes from the basic definition of what the integers are. However, for the number systems **Q**, **R** and **C** it can be proven that each of these systems are integral domains. There are, of course, other number systems (sets *S* with operations + and · which form commutative rings) which are not integral domains.

Give a proof that **R** with + and · is an integral domain, i.e., show using the basic properties of + and · for **R** that if *a*, *b* 0 **R** and $a \cdot b = 0$, then either $a = 0$ or $b = 0$. Give two examples, one infinite and one finite, of commutative rings with unity which are not an integral domains.

Writing Assignment #5 Due April 3

We have studied groups thus far. Of course, we are familiar now with the definition that a group is a set *G* with an operation * such that

(i) * is associative on *G*
(ii) there exists an (two-sided) identity element *e* in *G* relative to *
(iii) for each *a* in *G* there exists a (two-sided) inverse in *G*.

If a set *G* with operation * satisfies only (i), then it is called a *semi-group*. If a set *G* with operation * satisfies only (i) and (ii), then it is called a *monoid*.

Give one example of a semi-group. Give two examples of monoids, one infinite and one finite.

Theorem 5.1 and 14.1 describe general properties which hold for groups. Which of these properties still hold for monoids, i.e., which of these properties can be deduced without needing to appeal to the availability of an inverse?

Can the discussion of integral powers of an element and the "laws of exponents", which Durban pursues on the bottom half of page 79, be extended from the setting of groups to the setting of monoids? If not all of it, which part?

Writing Assignment #6 Due April 17

Let G be <u>any</u> finite group. Suppose that G has a subgroup H of index 2, i.e., when the group G is partitioned by H into right cosets there are only 2 such cosets.

Explain why the 2 right cosets of H behave like $\mathbf{Z}_2$, i.e., either (1) show that there is an arithmetic on the right cosets which is exactly like the arithmetic in $\mathbf{Z}_2$ (problem 17.25 may help) or (2) show that there is an isomorphism between the right cosets and $\mathbf{Z}_2$.

Writing Assignment #7 Due May 1

When we studied groups we introduced the idea of a subgroup and we found that not all subgroups share the same properties. We distinguished <u>normal</u> subgroups as those which have the same left and right cosets. We found that they are intimately related to group homomorphisms, i.e., we found that the kernel of every group homomorphism is a normal subgroup and every homomorphic image of a group is isomorphic to some quotient group modulo a normal subgroup (Fundamental Homomorphism Theorem for Groups).

We have now introduced rings and the idea of a subring. It can be observed that not all subrings share the same properties. The ring analog of a normal subgroup is called an <u>ideal</u>, i.e., an ideal shares properties analogous to those of a normal subgroup. Specifically, the kernel of every ring homomorphism is an ideal and every homomorphic image of a ring is isomorphic to some quotient ring modulo an ideal (Fundamental Homomorphism Theorem for Rings).

The following is the formal definition of an ideal.

Definition. A subring I of a ring R is an *ideal* if $ar \in I$ and $ra \in I$ for all $a \in I$ and for all $r \in R$.

In Section 45, the author observes (in Theorem 45.1) some of the results for subrings, ideals and ring homomorphisms which are analogous to those holding for groups. In Section 46, the author proves (in Theorem 46.1) that the set of right cosets of an ideal form a ring (called the quotioent ring and denoted R/I).

(a) Give an example of a subring which is an ideal. Give an example of a subring which is not an ideal.

(b) Prove Theorem 46.2