1. Let $S$ be the set of positive real numbers. Let $m*n = \sqrt{mn}$.
   a) Is $*$ an operation on $S$?
   b) If $*$ is an operation on $S$, is it commutative?
   c) If $*$ is an operation on $S$, is it associative?
   d) If $*$ is an operation on $S$, does there exist an identity for $*$?

2. Let $G = \{ 2^p : p \in \mathbb{Z} \}$. Let $*$ be multiplication. Is $(G,*)$ a group?

3. Consider the group $(\text{M}(2,\mathbb{Z}),+)$. Let $S = \{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a, b \in \mathbb{Z} \}$, $T = \{ \begin{bmatrix} a & b \\ 1 & 1 \end{bmatrix} : a, b \in \mathbb{Z} \}$.
   a) Is $(S, +)$ a subgroup of $(\text{M}(2,\mathbb{Z}),+)$?
   b) Is $(T, +)$ a subgroup of $(\text{M}(2,\mathbb{Z}),+)$?

4. Identify the symmetry group for the following figure, which consists of a square with one diagonal and the middle third of the other diagonal.

5. Let $n$ be a fixed positive integer. Show that if $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$, then $a \equiv c \pmod{n}$.

6. Prove or disprove: $(a, b) = 1$ and $(c, d) = 1$ implies $(ac, bd) = 1$.

7. Consider the group $(G, \oplus)$, where $G$ is the subset of $\mathbb{Z}_{42}$ given by $\langle [3] \rangle$. Determine whether the subset $S$ of $G$ given by $S = \{ [12], [24], [36], [48], [60], [72], [84] \}$ is a subgroup of $G$.

8. Find $(136, 26)$ and write it as a linear combination of 136 and 26.

9. Let $(G, \ast)$ be a group and let $A, B$ be subgroups of $G$. Prove or disprove the following:
   a) $A \cup B$ is again a subgroup.
   b) $A \cap B$ is again a subgroup.

10. Let $(G, \ast)$ be an Abelian group with identity $e$ and let $a, b \in G$. Let $o(a) = m$ and $o(b) = n$.
    a) Prove that $(ab)^{mn} = e$.
    b) How is $o(ab)$ related to $mn$?
11. Let $G = M_{(\cdot)}$ and let $H = \langle \mu_{270} \rangle$. Find $[G : H]$. 

12. Find the subgroup lattice for:
   a) $M_{(\cdot)}$
   b) $Z_{24}$

13. Let $\Theta$ be a group homomorphism, $\Theta : G \to H$, and let $a \in G$. Show that $\Theta(a^{-1}) = \Theta(a)^{-1}$. 

14. Determine whether the following pairs of groups, with addition as the operation in each case, are isomorphic or not (prove or disprove).
   a) $Z_5 \times Z_{10}$ and $Z_{50}$
   b) $Z$ and $E$, where $E$ is the set of even integers.

15. Find the smallest subring of $\mathbb{Q}$ which contains $\frac{1}{2}$. 

16. Verify that $Z_2 \times Z_3$ is not an integral domain. 

17. Let $D$ be a well-ordered integral domain. Let $a \in D$ and suppose that both $a \neq 0$ and $a \neq e$, where $e$ is the unity of $D$. Prove that $a^2 > a$. 

18. Consider $M(2, \mathbb{Z})$. Let $B = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$. Consider the mappings $\lambda$, $\mu$ as mappings from the additive group $M(2, \mathbb{Z})$ to the additive group $M(2, \mathbb{Z})$ where $\lambda$ is given by $\lambda(A) = BA$ for $A \in M(2, \mathbb{Z})$ and $\mu$ is given by $\mu(A) = AB$ for $A \in M(2, \mathbb{Z})$. Find $\text{ker}(\lambda)$ and $\text{ker}(\mu)$. 

19. For each of the following give an example which satisfies the condition if such an example exists; if no example exists, explain why.
   a) rational + rational = irrational 
   b) irrational + irrational = rational 
   c) irrational + rational = rational 
   d) irrational + rational = irrational 

20. For $z \in \mathbb{C}$ let $\Theta(z) = \bar{z}$. Show that $\Theta$ is a ring isomorphism of $\mathbb{C}$ to $\mathbb{C}$. 

21. Let $n$ be a fixed positive integer and $T_n$ be the set of $n^{th}$ complex roots of unity, i.e., $T_n = \{ z = \cos \left( \frac{2\pi k}{n} \right) + i \sin \left( \frac{2\pi k}{n} \right) : k \in \mathbb{Z} \}$. Show that $T_n$ with multiplication as its operation is a group. 

22. Let $G$ and $H$ be groups and let $\Theta$ be a homomorphism which maps $G$ onto $H$. Prove or disprove the following:
   a) $H$ is Abelian implies $G$ is Abelian. 
   b) $G$ is Abelian implies $H$ is Abelian.