MATH 3360 Final Exam May 8, 2000

Answer the problems on separate paper. You do <u>not</u> need to rewrite the problem statements on your answer sheets. Do your own work. <u>*Show*</u> all relevant steps which lead to your solutions. Retain this question sheet for your records.

Notation:

$\mathbf{Z} = \{ n : n \text{ is an integer} \}$

$\mathbf{Q} = \{ r : r \text{ is a rational number} \}$

$\mathbb{R} = \{ x : x \text{ is a real number} \}$

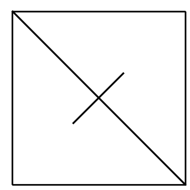
$\mathbb{C} = \{ z : z \text{ is a complex number} \}$

1. Let S be the set of positive real numbers. Let $m*n = \sqrt{mn}$.
   a) Is * an operation on S?
   b) If * is an operation on S, is it commutative?
   c) If * is an operation on S, is it associative?
   d) If * is an operation on S, does there exist an identity for *?

2. Let $G = \{ 2^p : p \in \mathbf{Z}\}$. Let * be multiplication. Is $(G,*)$ a group?

3. Consider the group $(M(2,\mathbf{Z}),+)$. Let $S = \{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a, b \in \mathbf{Z} \}$, $T = \{ \begin{bmatrix} a & b \\ 1 & 1 \end{bmatrix} : a, b \in \mathbf{Z} \}$.
   a) Is $(S,+)$ a subgroup of $(M(2,\mathbf{Z}),+)$?
   b) Is $(T,+)$ a subgroup of $(M(2,\mathbf{Z}),+)$?

4. Identify the symmetry group for the following figure, which consists of a square with one diagonal and the middle third of the other diagonal.



5. Let n be a fixed positive integer. Show that if $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$, then $a \equiv c \pmod{n}$.

6. Prove or disprove: $(a, b) = 1$ and $(c, d) = 1$ implies $(ac, bd) = 1$.

7. Consider the group $(G,\oplus)$, where G is the subset of $\mathbf{Z}_{42}$ given by $< [3] >$. Determine whether the subset S of G given by $S = \{[12], [24], [36], [48], [60], [72], [84]\}$ is a subgroup of G.

8. Find $(136,26)$ and write it as a linear combination of 136 and 26.

9. Let $(G,*)$ be a group and let A, B be subgroups of G. Prove or disprove the following:
   a) $A \cup B$ is again a subgroup.
   b) $A \cap B$ is again a subgroup.

10. Let $(G,*)$ be an Abelian group with identity e and let $a, b \in G$. Let $o(a) = m$ and $o(b) = n$.
    a) Prove that $(ab)^{mn} = e$.
    b) How is $o(ab)$ related to mn?

11. Let $G = \mathrm{M}_{(\square)}$ and let $H = < \mu_{270} >$. Find $[G : H]$.

12. Find the subgroup lattice for:
   a) $\mathrm{M}_{(\triangleright)}$.
   b) $\mathbf{Z}_{24}$

13. Let θ be a group homomorphism, $\theta : G \rightarrow H$, and let $a \in G$. Show that $\theta(a^{-1}) = \theta(a)^{-1}$.

14. Determine whether the following pairs of groups, with addition as the operation in each case, are isomorphic or not (prove or disprove).
   a) $\mathbf{Z}_5 \times \mathbf{Z}_{10}$ and $\mathbf{Z}_{50}$
   b) $\mathbf{Z}$ and $\mathbf{E}$, where $\mathbf{E}$ is the set of even integers.

15. Find the smallest subring of $\mathbf{Q}$ which contains ½.

16. Verify that $\mathbf{Z}_2 \times \mathbf{Z}_3$ is not an integral domain.

17. Let D be a well-ordered integral domain. Let $a \in D$ and suppose that both $a \neq 0$ and $a \neq e$, where e is the unity of D. Prove that $a^2 > a$.

18. Consider M(2,$\mathbf{Z}$). Let $B = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$. Consider the mappings λ, μ as mappings from the additive group M(2,$\mathbf{Z}$) to the additive group M(2,$\mathbf{Z}$) where λ is given by $\lambda(A) = BA$ for $A \in$ M(2,$\mathbf{Z}$) and μ is given by $\mu(A) = AB$ for $A \in$ M(2,$\mathbf{Z}$). Find ker(λ) and ker(μ).

19. For each of the following give an example which satisfies the condition if such an example exists; if no example exists, explain why.
   a) rational + rational = irrational
   b) irrational + irrational = rational
   c) irrational + rational = rational
   d) irrational + rational = irrational

20. For $z \in \mathbb{C}$ let $\theta(z) = \overline{z}$. Show that θ is a ring isomorphism of $\mathbb{C}$ to $\mathbb{C}$.

21. Let n be a fixed positive integer and T_n be the set of n[th] complex roots of unity, i.e., $T_n = \{z = \cos\left(\frac{2\pi k}{n}\right) + i \sin\left(\frac{2\pi k}{n}\right) : k \in \mathbf{Z}\}$. Show that T_n with multiplication as its operation is a group.

22. Let G and H be groups and let θ be a homomorphism which maps G onto H. Prove or disprove the following:
   a) H is Abelian implies G is Abelian.
   b) G is Abelian implies H is Abelian.