MATH 3360 Exam I February 16, 2000

Answer the problems on separate paper. You do not need to rewrite the problem statements on your answer sheets. Do your own work. *Show* all relevant steps which lead to your solutions. Retain this question sheet for your records.

$\mathbf{Z} = \{ n : n \text{ is an integer}\}$ $\mathbf{Q} = \{ r : r \text{ is a rational number} \}$

$\mathbf{Z}^+ = \{ n : n \in \mathbf{Z} \text{ and } n > 0 \}$ $\mathbf{Q}^+ = \{ r : r \in \mathbf{Q} \text{ and } r > 0 \}$

$\mathbb{R} = \{ x : x \text{ is a real number} \}$

1. [18pts.] Let $E$ be the set of even integers. Let $m*n = m + 3n$
   (a) Show that * is an operation on $E$.
   (b) Is * associative? Why or why not?
   (c) Does * have an identity? Why or why not?

2. [12pts.] Complete the Cayley table so that the operation * is commutative, has an identity and each element has an inverse.

| * | v | w | x | z |
|---|---|---|---|---|
| v | | | | |
| w | | | | x |
| x | | z | w | |
| z | | | | w |

3. [24pts.] Determine whether the set $G$ and the operation * form a group. If so, specify the identity. If not, state a reason why.

   a) $G = \mathbf{Q}^+$, * = (real) multiplication
   b) $G = \mathbf{Z}^+$, * = (real) addition
   c) $G = \{3^m : m \in \mathbf{Z} \}$, * is defined by $m*n = mn$.

4. [18pts.] Determine whether the given subset $H$ is a subgroup of the specified group $(G,*)$.

   a) $G = \{1, 2, 3, \ldots, 9, 10, 11, 12\}$, * = clock arithmetic
   $H = \{3, 6, 9, 12\}$
   b) $G = \{ \alpha_{(a,b)} \mid \alpha_{(a,b)}(n) = an + b, \quad a = 1, b \in \mathbf{Z} \}$, * = composition,
   $H = \{ \alpha_{(a,b)} \in G \mid b \text{ is an odd multiple of } 5 \}$

5. [12pts.] Write each of the following permutations as a single cycle or a product of pairwise disjoint cycles:

   a) ( 4 6 5)(1 2 5 3)(1 3 4)

   b) (4 2 3)(1 2 5 3)(1 3 4)

   c) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 7 & 5 & 4 & 1 & 8 & 6 & 2 \end{pmatrix}$

6. [16pts.] Let $\alpha : \mathbb{R} \to \mathbb{R}$ be given by $\alpha(x) = 3x + 1$, $\beta : \mathbb{R} \to \mathbb{R}$ be given by $\beta(x) = \arctan x$ and $\gamma : \mathbb{R} \to \mathbb{R}$ be given by $\gamma(x) = 1/(x^2 + 1)$. Determine which of the following are one-to-one? which are onto?

   a) $\alpha$ b) $\beta$ c) $\gamma$ d) $\beta\circ\alpha$