EXAMPLE 10 Higher-order partial derivatives of a function of several variables

By direct calculation, show that $f_{xyz} = f_{yzx} = f_{zyx}$ for the function $f(x, y, z) = xyz + x^2y^3z^4$.

Solution

First, compute the partials:

$$f_x(x, y, z) = yz + 2xy^3z^4$$
$$f_y(x, y, z) = xz + 3x^2y^2z^4$$
$$f_z(x, y, z) = xy + 4x^2y^3z^3$$

Next, determine the mixed partials:

$$f_{xy}(x, y, z) = (yz + 2xy^3z^4)_y = z + 6xy^2z^4$$
$$f_{yz}(x, y, z) = (xz + 3x^2y^2z^4)_z = x + 12x^2y^2z^3$$
$$f_{zy}(x, y, z) = (xy + 4x^2y^3z^3)_y = x + 12x^2y^2z^3$$

Finally, obtain the required higher mixed partials:

$$f_{xyz}(x, y, z) = (z + 6xy^2z^4)_z = 1 + 24xy^2z^3$$
$$f_{yzx}(x, y, z) = (x + 12x^2y^2z^3)_x = 1 + 24xy^2z^3$$
$$f_{zyx}(x, y, z) = (x + 12x^2y^2z^3)_x = 1 + 24xy^2z^3$$

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11.3 PROBLEM SET

A 1. **WHAT DOES THIS SAY?** What is a partial derivative?

2. **Exploration Problem** Describe two fundamental interpretations of the partial derivatives $f_x(x, y)$ and $f_y(x, y)$.

Determine f_x, f_y, f_{xx}, and f_{yx} in Problems 3–8.

3. $f(x, y) = x^3 + x^2y + xy^2 + y^3$
4. $f(x, y) = (x + xy + y)^3$
5. $f(x, y) = \dfrac{x}{y}$
6. $f(x, y) = xe^{xy}$
7. $f(x, y) = \ln(2x + 3y)$
8. $f(x, y) = \sin x^2y$

Determine f_x and f_y in Problems 9–16.

9. a. $f(x, y) = (\sin x^2)\cos y$ b. $f(x, y) = \sin(x^2 \cos y)$
10. a. $f(x, y) = (\sin\sqrt{x})\ln y^2$ b. $f(x, y) = \sin(\sqrt{x}\ln y^2)$
11. $f(x, y) = \sqrt{3x^2 + y^4}$
12. $f(x, y) = xy^2\ln(x + y)$
13. $f(x, y) = x^2e^{x+y}\cos y$
14. $f(x, y) = xy^3\tan^{-1} y$
15. $f(x, y) = \sin^{-1}(xy)$
16. $f(x, y) = \cos^{-1}(xy)$

Determine f_x, f_y, and f_z in Problems 17–22.

17. $f(x, y, z) = xy^2 + yz^3 + xyz$
18. $f(x, y, z) = xye^z$
19. $f(x, y, z) = \dfrac{x + y^2}{z}$
20. $f(x, y, z) = \dfrac{xy + yz}{xz}$
21. $f(x, y, z) = \ln(x + y^2 + z^3)$
22. $f(x, y, z) = \sin(xy + z)$

In Problems 23–28, determine $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ by differentiating implicitly.

23. $\dfrac{x^2}{9} - \dfrac{y^2}{4} + \dfrac{z^2}{2} = 1$
24. $3x^2 + 4y^2 + 2z^2 = 5$
25. $3x^2y + y^3z - z^2 = 1$
26. $x^3 - xy^2 + yz^2 - z^3 = 4$
27. $\sqrt{x} + y^2 + \sin xz = 2$
28. $\ln(xy + yz + xz) = 5 \quad (x > 0,\ \ y > 0,\ \ z > 0)$

In Problems 29–32, compute the slope of the tangent line to the graph of f at the given point P_0 in the direction parallel to

a. the xz-plane *b. the yz-plane*

29. $f(x, y) = xy^3 + x^3y$; $P_0(1, -1, -2)$
30. $f(x, y) = \dfrac{x^2 + y^2}{xy}$; $P_0(1, -1, -2)$
31. $f(x, y) = x^2\sin(x + y)$; $P_0\left(\frac{\pi}{2}, \frac{\pi}{2}, 0\right)$
32. $f(x, y) = x\ln(x + y^2)$; $P_0(e, 0, e)$

B 33. Determine f_x and f_y for

$$f(x, y) = \int_x^y (t^2 + 2t + 1)\,dt$$

Hint: Review the second fundamental theorem of calculus.