EXAMPLE 10 Using the binomial series theorem to obtain a Maclaurin series expansion

Determine the Maclaurin series for $f(x) = \sqrt{9+x}$ and find its interval of convergence.

Solution

Write $f(x) = \sqrt{9+x} = (9+x)^{1/2} = 3\left(1+\frac{x}{9}\right)^{1/2}$. Thus,

$$\begin{aligned}\sqrt{9+x} &= 3\left(1+\frac{x}{9}\right)^{1/2}\\ &= 3\left[1+\frac{1}{2}\left(\frac{x}{9}\right)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(\frac{x}{9}\right)^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}\left(\frac{x}{9}\right)^3+\cdots\right]\\ &= 3\left[1+\frac{1}{18}x-\frac{1}{648}x^2+\frac{1}{11,664}x^3-\cdots\right]\\ &= 3+\frac{1}{6}x-\frac{1}{216}x^2+\frac{1}{3,888}x^3-\cdots\end{aligned}$$

Since the exponent $p = \frac{1}{2}$ satisfies $p > 0$, p not an integer, it follows from Theorem 8.26 that the series converges for $\left|\frac{x}{9}\right| \le 1$, that is, for $|x| \le 9$. ■

8.8 PROBLEM SET

A

1. **WHAT DOES THIS SAY?** Compare and contrast Maclaurin and Taylor series.
2. **WHAT DOES THIS SAY?** Discuss the binomial series theorem.

Find the Maclaurin series for the functions given in Problems 3–30. Assume that a is any constant, and that all the derivatives of all orders exist at x = 0.

3. e^{2x}
4. e^{-x}
5. e^{x^2}
6. e^{ax}
7. $\sin x^2$
8. $\sin^2 x$
9. $\sin ax$
10. $\cos ax$
11. $\cos 2x^2$
12. $\cos x^3$
13. $x^2 \cos x$
14. $\sin \frac{x}{2}$
15. $x^2 + 2x + 1$
16. $x^3 - 2x^2 + x - 5$
17. xe^x
18. $e^{-x} + e^{2x}$
19. $e^x + \sin x$
20. $\sin x + \cos x$
21. $\frac{1}{1+4x}$
22. $\frac{1}{1-ax}$, $a \ne 0$
23. $\frac{1}{a+x}$, $a \ne 0$
24. $\frac{1}{a^2+x^2}$, $a \ne 0$
25. $\ln(3+x)$
26. $\log(1+x)$
27. $\tan^{-1}(2x)$
28. $\sqrt{1-x}$
29. e^{-x^2}
30. $f(x) = \begin{cases}\frac{\sin x}{x} & \text{if } x \ne 0\\ 1 & \text{if } x = 0\end{cases}$

Find the first four terms of the Taylor series of the functions in Problems 31–42 at the given value of c.

31. $f(x) = e^x$ at $c = 1$
32. $f(x) = \ln x$ at $c = 3$
33. $f(x) = \cos x$ at $c = \frac{\pi}{3}$
34. $f(x) = \sin x$ at $c = \frac{\pi}{4}$
35. $f(x) = \tan x$ at $c = 0$
36. $f(x) = x^2 + 2x + 1$ at $c = 200$
37. $f(x) = x^3 - 2x^2 + x - 5$ at $c = 2$
38. $f(x) = \sqrt{x}$ at $c = 9$
39. $f(x) = \frac{1}{2-x}$ at $c = 5$
40. $f(x) = \frac{1}{4-x}$ at $c = -2$
41. $f(x) = \frac{3}{2x-1}$ at $c = 2$
42. $f(x) = \frac{5}{3x+2}$ at $c = 2$

Expand each function in Problems 43–48 as a binomial series. Give the interval of convergence of the series.

43. $f(x) = \sqrt{1+x}$
44. $f(x) = \frac{1}{\sqrt{1+x^2}}$
45. $f(x) = (1+x)^{2/3}$
46. $f(x) = (4+x)^{-1/3}$
47. $f(x) = \frac{x}{\sqrt{1-x^2}}$
48. $f(x) = \sqrt[4]{2-x}$

B

49. Use term-by-term integration to show that
$$\ln(1+x) = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}x^k}{k} \quad \text{for } |x| < 1$$
50. Use the Maclaurin series for e^x and e^{-x} to find the Maclaurin series for
$$\cosh x = \frac{e^x + e^{-x}}{2}$$
51. Use the Maclaurin series for e^x and e^{-x} to find the Maclaurin series for
$$\sinh x = \frac{e^x - e^{-x}}{2}$$