If we multiply both sides by 4, we find another series for π:

$$\pi = 16\left[\tfrac{1}{5} - \tfrac{1}{3}\left(\tfrac{1}{5}\right)^3 + \tfrac{1}{5}\left(\tfrac{1}{5}\right)^5 - \cdots\right]$$
$$- 4\left[\tfrac{1}{239} - \tfrac{1}{3}\left(\tfrac{1}{239}\right)^3 + \tfrac{1}{5}\left(\tfrac{1}{239}\right)^5 - \cdots\right]$$

By comparing the spreadsheet in the margin with the one in Figure 8.22, we see that Machin's formula for π converges much more rapidly than the Leibniz formula. Indeed, the sum of the first seven terms $M_7 = 3.141592654$ already coincides with π for the first eight decimal places.

8.7 PROBLEM SET

A *Find the convergence set for the power series given in Problems 1–28.*

1. $\sum_{k=1}^{\infty} \frac{kx^k}{k+1}$
2. $\sum_{k=1}^{\infty} \frac{k^2x^k}{k+1}$
3. $\sum_{k=1}^{\infty} \frac{k(k+1)x^k}{k+2}$
4. $\sum_{k=1}^{\infty} \sqrt{k-1}\,x^k$
5. $\sum_{k=1}^{\infty} k^2 3^k (x-3)^k$
6. $\sum_{k=1}^{\infty} \frac{k^2(x-2)^k}{3^k}$
7. $\sum_{k=0}^{\infty} \frac{3^k(x+3)^k}{4^k}$
8. $\sum_{k=0}^{\infty} \frac{4^k(x+1)^k}{3^k}$
9. $\sum_{k=0}^{\infty} \frac{k!(x-1)^k}{5^k}$
10. $\sum_{k=0}^{\infty} \frac{(x-15)^k}{\ln(k+1)}$
11. $\sum_{k=1}^{\infty} \frac{k^2}{2^k}(x-1)^k$
12. $\sum_{k=1}^{\infty} \frac{2^k(x-3)^k}{k(k+1)}$
13. $\sum_{k=1}^{\infty} \frac{k(3x-4)^k}{(k+1)^2}$
14. $\sum_{k=0}^{\infty} \frac{(2x+3)^k}{4^k}$
15. $\sum_{k=1}^{\infty} \frac{kx^k}{7^k}$
16. $\sum_{k=0}^{\infty} \frac{(2k)!x^k}{(3k)!}$
17. $\sum_{k=1}^{\infty} \frac{(k!)^2x^k}{k^k}$
18. $\sum_{k=1}^{\infty} \frac{(-1)^k kx^k}{\ln(k+2)}$
19. $\sum_{k=2}^{\infty} \frac{(-1)^k x^k}{k(\ln k)^2}$
20. $\sum_{k=0}^{\infty} \frac{(3x)^k}{2^{k+1}}$
21. $\sum_{k=0}^{\infty} \frac{(2x)^{2k}}{k!}$
22. $\sum_{k=0}^{\infty} \frac{(x+2)^{2k}}{3^k}$
23. $\sum_{k=0}^{\infty} \frac{k!}{2^k}(3x)^{3k}$
24. $\sum_{k=1}^{\infty} \frac{(3x)^{3k}}{\sqrt{k}}$
25. $\sum_{k=0}^{\infty} \frac{2^k}{k!}(2x-1)^{2k}$
26. $\sum_{k=0}^{\infty} 2^k(3x)^{3k}$
27. $\sum_{k=1}^{\infty} \frac{x^k}{k\sqrt{k}}$
28. $\sum_{k=1}^{\infty} \frac{(\ln k)x^k}{k}$

B *Find the radius of convergence R in Problems 29–34.*

29. $\sum_{k=1}^{\infty} k^2(x+1)^{2k+1}$
30. $\sum_{k=1}^{\infty} 2^{\sqrt{k}}(x-1)^k$
31. $\sum_{k=1}^{\infty} \frac{k!x^k}{k^k}$
32. $\sum_{k=1}^{\infty} \frac{(k!)^2x^k}{(2k)!}$
33. $\sum_{k=1}^{\infty} k(ax)^k$ for constant a
34. $\sum_{k=1}^{\infty} (a^2x)^k$ for constant a

In Problems 35–38, find the derivative $f'(x)$ by differentiating term by term.

35. $f(x) = \sum_{k=0}^{\infty} \left(\frac{x}{2}\right)^k$
36. $f(x) = \sum_{k=1}^{\infty} \frac{x^k}{k}$
37. $f(x) = \sum_{k=0}^{\infty} (k+2)x^k$
38. $f(x) = \sum_{k=0}^{\infty} kx^k$

In Problems 39–42, find $\int_0^x f(u)\,du$ by integrating term by term.

39. $f(x) = \sum_{k=0}^{\infty} \left(\frac{x}{2}\right)^k$
40. $f(x) = \sum_{k=1}^{\infty} \frac{x^k}{k}$
41. $f(x) = \sum_{k=0}^{\infty} (k+2)x^k$
42. $f(x) = \sum_{k=1}^{\infty} kx^k$

43. **Counterexample Problem** Show that the series

$$S = \sum_{k=1}^{\infty} \frac{\sin(k!x)}{k^2}$$

converges for all x. Differentiate term by term to obtain the series

$$T = \sum_{k=1}^{\infty} \frac{k!\cos(k!x)}{k^2}$$

Show that this series diverges for all x. Why does this not violate Theorem 8.23?

C 44. Suppose $\{a_k\}$ is a sequence for which

$$\lim_{k\to\infty} \sqrt[k]{|a_k|} = \frac{1}{R}$$

Show that the power series

$$\sum_{k=1}^{\infty} ka_kx^{k-1}$$

has radius of convergence R.