Solution

The general term $a_k = \dfrac{k!}{k^k}$ involves a kth power, which suggests using the root test, but the presence of the factorial $k!$ makes using the ratio test much more reasonable.

$$L = \lim_{k\to\infty} \frac{\dfrac{(k+1)!}{(k+1)^{k+1}}}{\dfrac{k!}{k^k}} = \lim_{k\to\infty} \frac{(k+1)!k^k}{k!(k+1)^{k+1}} = \lim_{k\to\infty} \frac{(k+1)k^k}{(k+1)^{k+1}}$$

$$= \lim_{k\to\infty} \left(\frac{k}{k+1}\right)^k = \lim_{k\to\infty} \frac{1}{\left(1+\dfrac{1}{k}\right)^k} = \frac{1}{e} < 1$$

Since $L < 1$, the series converges.

8.5 PROBLEM SET

A

1. **WHAT DOES THIS SAY?** Describe the ratio test.
2. **WHAT DOES THIS SAY?** Describe the root test.

Use either the ratio test or the root test to determine the convergence of the series given in Problems 3–26.

3. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k!}$
4. $\displaystyle\sum_{k=1}^{\infty} \frac{k!}{2^k}$
5. $\displaystyle\sum_{k=1}^{\infty} \frac{k!}{2^{3k}}$
6. $\displaystyle\sum_{k=1}^{\infty} \frac{3^k}{k!}$
7. $\displaystyle\sum_{k=1}^{\infty} \frac{k}{2^k}$
8. $\displaystyle\sum_{k=1}^{\infty} \frac{2^k}{k^2}$
9. $\displaystyle\sum_{k=1}^{\infty} \frac{k^{100}}{e^k}$
10. $\displaystyle\sum_{k=1}^{\infty} ke^{-k}$
11. $\displaystyle\sum_{k=1}^{\infty} k\left(\frac{4}{3}\right)^k$
12. $\displaystyle\sum_{k=1}^{\infty} k\left(\frac{3}{4}\right)^k$
13. $\displaystyle\sum_{k=1}^{\infty} \left(\frac{2}{k}\right)^k$
14. $\displaystyle\sum_{k=1}^{\infty} \frac{k^{10}2^k}{k!}$
15. $\displaystyle\sum_{k=1}^{\infty} \frac{k^5}{10^k}$
16. $\displaystyle\sum_{k=1}^{\infty} \frac{3^k}{k^2}$
17. $\displaystyle\sum_{k=1}^{\infty} \left(\frac{k}{3k+1}\right)^k$
18. $\displaystyle\sum_{k=1}^{\infty} \frac{3k+1}{2^k}$
19. $\displaystyle\sum_{k=1}^{\infty} \frac{k!}{(k+2)^4}$
20. $\displaystyle\sum_{k=1}^{\infty} \frac{k^5+100}{k!}$
21. $\displaystyle\sum_{k=1}^{\infty} \frac{(k!)^2}{(2k)!}$
22. $\displaystyle\sum_{k=1}^{\infty} \frac{(k!)^2}{[(2k)!]^2}$
23. $\displaystyle\sum_{k=1}^{\infty} k^2 2^{-k}$
24. $\displaystyle\sum_{k=1}^{\infty} k^4 3^{-k}$
25. $\displaystyle\sum_{k=1}^{\infty} \left(\frac{k-2}{k}\right)^{k^2}$
26. $\displaystyle\sum_{k=1}^{\infty} \left(\frac{k}{2k+1}\right)^k$

Test the series in Problems 27–44 for convergence. Justify your answers (that is, state explicitly which test you are using).

27. $\displaystyle\sum_{k=1}^{\infty} \frac{1{,}000}{k}$
28. $\displaystyle\sum_{k=1}^{\infty} \frac{5{,}000}{k\sqrt{k}}$
29. $\displaystyle\sum_{k=1}^{\infty} \frac{5k+2}{k2^k}$
30. $\displaystyle\sum_{k=1}^{\infty} \frac{(k!)^2}{k^k}$
31. $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt{k!}}{2^k}$
32. $\displaystyle\sum_{k=1}^{\infty} \frac{3k+5}{k3^k}$
33. $\displaystyle\sum_{k=1}^{\infty} \frac{2^k k!}{k^k}$
34. $\displaystyle\sum_{k=1}^{\infty} \frac{2^{2k}k!}{k^k}$
35. $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt{k+1}}{k^{k+0.5}}$
36. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^k}$
37. $\displaystyle\sum_{k=1}^{\infty} \frac{k!}{(k+1)!}$
38. $\displaystyle\sum_{k=1}^{\infty} \frac{2^{1.000k}}{k^{k/2}}$
39. $\displaystyle\sum_{k=1}^{\infty} \left(1+\frac{1}{k}\right)^{-k^2}$
40. $\displaystyle\sum_{k=1}^{\infty} \left(\frac{k+2}{k}\right)^{-k^2}$
41. $\displaystyle\sum_{k=1}^{\infty} \left|\frac{\cos k}{2^k}\right|$
42. $\displaystyle\sum_{k=1}^{\infty} \left|\frac{\sin k}{3^k}\right|$
43. $\displaystyle\sum_{k=2}^{\infty} \left(\frac{\ln k}{k}\right)^k$
44. $\displaystyle\sum_{k=2}^{\infty} \frac{1}{(\ln k)^k}$

B *In Problems 45–52, assume $x > 0$ and find all x for which the given series converges.*

45. $\displaystyle\sum_{k=1}^{\infty} k^2 x^k$
46. $\displaystyle\sum_{k=1}^{\infty} kx^k$
47. $\displaystyle\sum_{k=1}^{\infty} \frac{(x+0.5)^k}{k\sqrt{k}}$
48. $\displaystyle\sum_{k=1}^{\infty} \frac{(3x+0.4)^k}{k^2}$
49. $\displaystyle\sum_{k=1}^{\infty} \frac{x^k}{k!}$
50. $\displaystyle\sum_{k=1}^{\infty} \frac{x^{2k}}{k}$