$$= 3 \lim_{t \to 1^-} [(t-1)^{1/3} - (-1)] = 3$$

That is, the improper integral converges and has the value 3. ■

EXAMPLE 9 Improper integral at a left endpoint

Find $\int_{\pi/2}^{\pi} \sec x \, dx$.

Solution

Because $\sec x$ is unbounded at the left endpoint $\frac{\pi}{2}$ of the interval of integration and is continuous on $[t, \pi]$ for any t with $\frac{\pi}{2} < t \leq \pi$, we find that

$$\int_{\pi/2}^{\pi} \sec x \, dx = \lim_{t \to (\pi/2)^+} \int_t^{\pi} \sec x \, dx = \lim_{t \to (\pi/2)^+} \ln |\sec x + \tan x| \Big|_t^{\pi}$$

$$= \lim_{t \to (\pi/2)^+} [\ln |-1 + 0| - \ln |\sec t + \tan t|] = -\infty$$

Thus, the integral diverges. ■

EXAMPLE 10 Improper integral at an interior point

Find $\int_0^3 (x-2)^{-1} \, dx$.

Solution

The integral is improper because the integrand is unbounded at $x = 2$. If the improper integral converges, we have

$$\int_0^3 (x-2)^{-1} \, dx = \int_0^2 (x-2)^{-1} \, dx + \int_2^3 (x-2)^{-1} \, dx$$

$$= \lim_{t \to 2^-} \int_0^t (x-2)^{-1} \, dx + \lim_{t \to 2^+} \int_t^3 (x-2)^{-1} \, dx$$

If either of these limits fails to exist, then the original integral diverges. Because

$$\lim_{t \to 2^-} \int_0^t (x-2)^{-1} \, dx = \lim_{t \to 2^-} \ln |x-2| \Big|_0^t$$

$$= \lim_{t \to 2^-} [\ln |t-2| - \ln 2]$$

$$= -\infty$$

we find that the original integral diverges. ■

WARNING A common mistake is to fail to notice the discontinuity at $x = 2$. It is WRONG to write

$$\int_0^3 (x-2)^{-1} \, dx = \ln |x-2| \Big|_0^3$$

$$= \ln 1 - \ln 2$$

$$= -\ln 2$$

Notice that this mistake leads to the conclusion that the integral converges, which, as you can see, is incorrect. You must also be cautious in using computer software with improper integrals, because it may not detect that the integral is improper.

7.7 PROBLEM SET

1. **WHAT DOES THIS SAY?** What is an improper integral?

2. **WHAT DOES THIS SAY?** Discuss the different types of improper integrals.

In Problems 3–46, either show that the improper integral converges and find its value, or show that it diverges.

3. $\int_1^{+\infty} \frac{dx}{x^3}$

4. $\int_1^{+\infty} \frac{dx}{\sqrt[3]{x}}$

5. $\int_1^{+\infty} \frac{dx}{x^{0.99}}$

6. $\int_1^{+\infty} \frac{dx}{\sqrt{x}}$

7. $\int_1^{+\infty} \frac{dx}{x^{1.1}}$

8. $\int_1^{+\infty} x^{-2/3} \, dx$

9. $\int_3^{+\infty} \frac{dx}{2x-1}$

10. $\int_3^{+\infty} \frac{dx}{\sqrt[3]{2x-1}}$

11. $\int_3^{+\infty} \frac{dx}{(2x-1)^2}$

12. $\int_0^{+\infty} e^{-x} \, dx$

13. $\int_0^{+\infty} 5e^{-2x} \, dx$

14. $\int_1^{+\infty} e^{1-x} \, dx$