This integral can be handled by the method of partial fractions.

$$\frac{-2}{(3u-1)(u+3)} = \frac{A_1}{3u-1} + \frac{A_2}{u+3}$$

Solve this to find $A_1 = -\frac{3}{5}$ and $A_2 = \frac{1}{5}$. We continue with the integration.

$$\begin{aligned}\int \frac{dx}{3\cos x - 4\sin x} &= \int \frac{-2\,du}{(3u-1)(u+3)}\\ &= \int \frac{-\frac{3}{5}\,du}{3u-1} + \int \frac{\frac{1}{5}\,du}{u+3}\\ &= -\frac{3}{5}\cdot\frac{1}{3}\ln|3u-1| + \frac{1}{5}\cdot\ln|u+3| + C\\ &= -\frac{1}{5}\ln\left|3\tan\frac{x}{2} - 1\right| + \frac{1}{5}\ln\left|\tan\frac{x}{2} + 3\right| + C\end{aligned}$$

■

SMH

Once again, observe that when carrying out integration, you may obtain very different forms for the result. For Example 9, you might use an integration table (Formula 393 in the *Student Mathematics Handbook*, for example) to find

$$\int \frac{du}{p\sin au + q\cos au} = \frac{1}{a\sqrt{p^2+q^2}}\ln\left|\tan\left(\frac{au + \tan^{-1}\left(\frac{q}{p}\right)}{2}\right)\right|$$

Let $p = -4$, $q = 3$, $a = 1$, so that

$$\int \frac{dx}{3\cos x - 4\sin x} = \frac{1}{5}\ln\left|\tan\left(\frac{x + \tan^{-1}\left(-\frac{3}{4}\right)}{2}\right)\right| + C$$

Problem 67 asks you to derive the formula

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C$$

from scratch. You might recall that we derived this formula using an unusual algebraic step in Example 3 of Section 7.1. You can now derive it by using a Weierstrass substitution.

7.4 PROBLEM SET

Write each rational function given in Problems 1–14 as a sum of partial fractions.

1. $\frac{1}{x(x-3)}$
2. $\frac{3x-1}{x^2-1}$
3. $\frac{3x^2+2x-1}{x(x+1)}$
4. $\frac{2x^2+5x-1}{x(x^2-1)}$
5. $\frac{4}{2x^2+x}$
6. $\frac{x^2-x+3}{x^2(x-1)}$
7. $\frac{4x^3+4x^2+x-1}{x^2(x+1)^2}$
8. $\frac{x^2-5x-4}{(x^2+1)(x-3)}$
9. $\frac{x^3+3x^2+3x-4}{x^2(x+3)^2}$
10. $\frac{1}{x^3-1}$
11. $\frac{1}{1-x^4}$
12. $\frac{x^4-x^2+2}{x^2(x-1)}$
13. $\frac{x^2+x-1}{x(x+1)(2x-1)}$
14. $\frac{x^3-2x^2+x-5}{x(x^2-1)(3x+5)}$

Compute the integrals given in Problems 15–20. Notice that in each case, the integrand is a rational function decomposed into partial fractions in Problems 1–6.

15. $\int \frac{dx}{x(x-3)}$
16. $\int \frac{3x-1}{x^2-1}\,dx$
17. $\int \frac{3x^2+2x-1}{x(x+1)}\,dx$
18. $\int \frac{2x^2+5x-1}{x(x^2-1)}\,dx$
19. $\int \frac{4\,dx}{2x^2+x}$
20. $\int \frac{x^2-x+3}{x^2(x-1)}\,dx$