Solution

This is similar to Formula 170, but you must take care of the 5 (constant multiple) and the 3 (by making a substitution).

Integrals Involving $\sqrt{u^2 + a^2}$

168. $\displaystyle\int \sqrt{u^2 + a^2}\, du = \frac{u\sqrt{u^2 + a^2}}{2} + \frac{a^2}{2} \ln\left|u + \sqrt{u^2 + a^2}\right|$

169. $\displaystyle\int u\sqrt{u^2 + a^2}\, du = \frac{(u^2 + a^2)^{3/2}}{3}$

170. $\displaystyle\int u^2\sqrt{u^2 + a^2}\, du = \frac{u(u^2 + a^2)^{3/2}}{4} - \frac{a^2 u\sqrt{u^2 + a^2}}{8} - \frac{a^4}{8} \ln\left|u + \sqrt{u^2 + a^2}\right|$

Let $u = \sqrt{3}x$; $du = \sqrt{3}\,dx$

$$\int 5x^2\sqrt{3x^2 + 1}\, dx = 5\int \left(\frac{u^2}{3}\right)\sqrt{u^2 + 1}\,\frac{du}{\sqrt{3}}$$

$u^2 = 3x^2$ so that $x^2 = \dfrac{u^2}{3}$

$$= \frac{5}{3\sqrt{3}}\int u^2\sqrt{u^2 + 1}\, du \qquad \text{Use Formula 170, where } a = 1.$$

$$= \frac{5}{3\sqrt{3}}\left[\frac{u(u^2 + 1)^{3/2}}{4} - \frac{u\sqrt{u^2 + 1}}{8} - \frac{1}{8}\ln\left|u + \sqrt{u^2 + 1}\right|\right] + C$$

$$= \frac{5}{24\sqrt{3}}\left[2\sqrt{3}\,x(3x^2 + 1)^{3/2} - \sqrt{3}\,x\sqrt{3x^2 + 1} - \ln\left|\sqrt{3}\,x + \sqrt{3x^2 + 1}\right|\right] + C$$

$$= \frac{5}{24}\left[2x(3x^2 + 1)^{3/2} - x\sqrt{3x^2 + 1} - \frac{1}{\sqrt{3}}\ln\left(\sqrt{3}\,x + \sqrt{3x^2 + 1}\right)\right] + C$$

You might want to show that $u + \sqrt{u^2 + 1} > 0$.

If you use a calculator or computer, you will probably obtain an alternate, but equivalent, form:

$$-\frac{5\sqrt{3}\ln\left|\sqrt{3x^2 + 1} + \sqrt{3}\,x\right|}{72} - \frac{5x\sqrt{3x^2 + 1}\,(6x^2 + 1)}{24}$$

■

7.1 PROBLEM SET

Find each integral in Problems 1–12.

1. $\displaystyle\int \frac{2x + 5}{\sqrt{x^2 + 5x}}\, dx$

2. $\displaystyle\int \frac{\ln x}{x}\, dx$

3. $\displaystyle\int \frac{dx}{x \ln x}$

4. $\displaystyle\int \cos x\, e^{\sin x}\, dx$

5. $\displaystyle\int \frac{x\, dx}{4 + x^4}$

6. $\displaystyle\int \frac{t^2\, dt}{9 + t^6}$

7. $\displaystyle\int (1 + \cot x)^4 \csc^2 x\, dx$

8. $\displaystyle\int \frac{4x^3 - 4x}{x^4 - 2x^2 + 3}\, dx$

9. $\displaystyle\int \frac{x^3 - x}{(x^4 - 2x^2 + 3)^2}\, dx$

10. $\displaystyle\int \frac{2x + 4}{x^2 + 4x + 3}\, dx$

11. $\displaystyle\int \frac{2x + 1}{x^2 + x + 1}\, dx$

12. $\displaystyle\int \frac{2x - 1}{(4x^2 - 4x)^2}\, dx$

Integrate the expressions in Problems 13–24 using the short table of integrals given in Appendix D.

13. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2 - a^2}}$

14. $\displaystyle\int \frac{dx}{x^2\sqrt{a^2 - x^2}}$

15. $\displaystyle\int x \ln x\, dx$

16. $\displaystyle\int \ln x\, dx$

17. $\displaystyle\int xe^{ax}\, dx$

18. $\displaystyle\int \frac{dx}{a + be^{2x}}$