Section I For each of the problems in the this section at least one of the choices is correct. For some of the problems more than one of the choices is correct. Record your answers for problems in this section on the answer sheet (last page).

1. (6 pts) Consider the following matrix $A=\begin{bmatrix} 1 & -2 & 0 & 3 \\ -2 & 1 & 3 & 3 \\ -1 & 2 & 1 & 2 \end{bmatrix}$. Which of the following matrices are row-equivalent to *A*.

a. $\begin{bmatrix} 1 & -2 & 0 & 3 \\ -2 & 1 & 3 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ b. $\begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ c. $\begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}$

d. $\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ e. $\begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & -3 & 3 & 9 \\ -1 & 2 & 1 & 2 \end{bmatrix}$ f. $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

2. (8 pts) Consider the following four matrices:

$$A=\begin{bmatrix} -3 & 0 & 1 \\ 1 & 2 & -1 \end{bmatrix} \quad B=\begin{bmatrix} 2 & 7 & -1 \\ 3 & -4 & 0 \end{bmatrix} \quad C=\begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix} \quad D=\begin{bmatrix} -1 & 0 \\ 2 & -3 \\ 1 & 1 \end{bmatrix}$$

Which of the following arithmetic operations are possible?

a. $3A - 2B$ b. AB c. $AC - D$ d. B^2

e. C^2 f. $(DA)^2$ g. D^2A^2 h. $DB - BD$

3. (8 pts) Consider the following subsets of $\mathbb{R}^3$:

a. $\{(x_1, x_2, x_3)^T \mid x_1 > 0 \text{ or } x_2 > 0\}$ b. $\{(x_1, x_2, x_3)^T \mid (x_1 = 0) \text{ or } (x_2 = 0) \text{ or } (x_3 = 0)\}$.

c. $\{(x_1, x_2, x_3)^T \mid x_1 - x_3 = 0\}$ d. $\{(x_1, x_2, x_3)^T \mid x_1^2 + x_2^2 + x_3^2 = 1\}$

Which of the above subsets of $\mathbb{R}^3$ are subspaces of $\mathbb{R}^3$?

4. (8 pts) Consider the following subsets of P_4:

a. $\{p \in P_4 \mid p(1) = 0\}$

b. $\{p \in P_4 \mid p(1) \geq p(0)\}$

c. $\{p \in P_4 \mid p \text{ is even}\}$

d. $\{p \in P_4 \mid \deg(p) = odd\}$

Which of the above subsets of P_4 are subspaces of P_4?

5. (8 pts) Consider the following subsets of $\mathbb{R}^3$:

a. $\left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} \right\}$

b. $\left\{ \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix} \right\}$

c. $\left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ -6 \\ 4 \end{bmatrix} \right\}$

d. $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \right\}$

Which of the above subsets of $\mathbb{R}^3$ are spanning sets for $\mathbb{R}^3$?

6. (8 pts) Consider the following subsets of $\mathbb{R}^3$:

a. $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right\}$

b. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -6 \\ -2 \\ 4 \end{bmatrix} \right\}$

c. $\left\{ \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \right\}$

d. $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \right\}$

Which of the above subsets of $\mathbb{R}^3$ are linearly independent?

7. (4 pts) Consider the set $S = \{1 + x, 1 - x, 1 - 2x + x^2\}$ which is a subset of P_3. Is the set S linear independent?

8. (8 pts) Consider the following subsets of $\mathbb{R}^3$:

a. $\left\{ \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$

b. $\left\{ \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} -5 \\ 1 \\ -4 \end{bmatrix} \right\}$

c. $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ -6 \end{bmatrix} \right\}$

d. $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$

Which of the above subsets of $\mathbb{R}^3$ forms a basis for $\mathbb{R}^3$?

9. (9 pts) Determine whether the following are linear transformations from $\mathbb{R}^4$ to $\mathbb{R}^3$

a. $L(\boldsymbol{x}) = \begin{bmatrix} x_1 - 4x_2 + 4x_4 \\ x_1 x_2 \\ -2x_1 + 2x_2 + x_3 \end{bmatrix}$

b. $L(\boldsymbol{x}) = \begin{bmatrix} x_4 - 1/x_2 \\ 0 \\ x_4 + 1/x \end{bmatrix}$

c. $L(\boldsymbol{x}) = \begin{bmatrix} x_1 \\ x_1 - x_2 \\ x_1 - x_2 + x_3 \end{bmatrix}$

10. (9 pts) Determine whether the following are linear transformations from P_3 to P_4

a. $L(p(x)) = x^2 p'(x)$

b. $L(p(x)) = \int_0^x p(t)\, dt$

c. $L(p(x)) = x(1 + p(x))$

11. (6 pts) Consider the sets of vectors given below. Which sets are orthogonal sets?

a. $\left\{ \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$

b. $\left\{ \begin{bmatrix} -1 \\ 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -3 \\ 2 \\ -5 \end{bmatrix} \right\}$

c. $\left\{ [1, \ -2, \ -1, \ 0, \ 3, \ -1]^T, [12, \ 3, \ 4, \ -4, \ -2, \ -4]^T \right\}$

12. (8pts) Which of the following sets forms an orthonormal basis for $\mathbb{R}^2$?

a. $\left\{\frac{1}{\sqrt{5}}[1,-2]^T, \frac{1}{\sqrt{5}}[2,1]^T\right\}$ b. $\left\{[3/5,-4/5]^T, [3/5,4/5]^T\right\}$

c. $\left\{\left[\sqrt{1}/2,-1/\sqrt{2}\right]^T, \left[1/\sqrt{2},1/\sqrt{2}\right]^T\right\}$ d. $\left\{[-1,0]^T, [0,-1]^T\right\}$

13. (9 pts) Let A be a 4×4 matrix. Which the following are (always) true?

a. $\dim(N(A)) = \dim(R(A^T))$ b. $\dim(N(A)) = \dim(R(A))$ c. $\dim(R(A)) = \dim(R(A^T))$

d. $N(A) \perp R(A^T)$ e. $N(A) \perp R(A)$ f. $R(A) \perp R(A^T)$

g. $\mathbb{R}^4 = N(A) \oplus R(A^T)$ h. $\mathbb{R}^4 = N(A) \oplus R(A)$ i. $\mathbb{R}^4 = R(A) \oplus R(A^T)$

Section II Answer the problems in this section on separate paper. You do not need to rewrite the problem statements on your answer sheets. Work carefully. Do your own work.

Show all relevant supporting steps!

14. (9 pts) Each of the following augmented matrices is in row echelon form.

i. For each matrix, indicate whether the corresponding system of linear equations is consistent or inconsistent

ii. For each case in which the corresponding system of linear equations is consistent, indicate whether the system has a unique solution or infinitely many solutions.

iii. For each case in which the corresponding system of linear equations is consistent and has a unique solution, find that unique solution.

a. $\left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$ b. $\left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$ c. $\left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

15. (8 pts) The following augmented matrix is in reduced row echelon form. Find the solution set of the corresponding system of linear equations.

a. $\left[\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

16. (9 pts) For each of the following pairs of matrices find an elementary matrix E such that $EA = B$.

a. $A = \begin{bmatrix} -1 & 0 & 2 \\ 1 & -1 & -1 \\ 2 & -1 & 2 \end{bmatrix}$ $B = \begin{bmatrix} -1 & 0 & 2 \\ 2 & -1 & 2 \\ 1 & -1 & -1 \end{bmatrix}$

b. $A = \begin{bmatrix} -1 & 2 & 0 & -2 \\ 2 & 2 & -1 & 3 \\ 1 & 2 & -1 & 4 \\ 0 & -1 & 2 & -3 \end{bmatrix}$ $B = \begin{bmatrix} -1 & 2 & 0 & -2 \\ 2 & 0 & 3 & -3 \\ 1 & 2 & -1 & 4 \\ 0 & -1 & 2 & -3 \end{bmatrix}$

17. (9 pts) Consider the matrix A given by $A = \begin{bmatrix} 1 & -2 & 0 & 2 & -2 & -1 \\ -2 & 1 & -1 & -1 & -3 & 2 \\ -1 & 2 & -1 & 0 & 2 & -1 \\ -2 & -2 & -3 & 4 & -9 & 0 \end{bmatrix}$. A straightforward reduction by elimination shows that A is row equivalent to U where

$$U = \begin{bmatrix} 1 & 0 & 0 & 4/3 & 0 & -7/3 \\ 0 & 1 & 0 & -1/3 & 0 & -2/3 \\ 0 & 0 & 1 & -2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

a. Find a basis for the row space of A.
b. Find a basis for the column space of A.
c. Find a basis for the null space of A.

18. (8 pts) Consider the linear transformation mapping $\mathbb{R}^3$ to $\mathbb{R}^3$ given by

$$L(\boldsymbol{x}) = (x_1 - 2x_2 - 3x_3, x_1 - x_2, x_1 + 3x_3)^T$$

a. Find the kernel of L.
b. Find the dimension of the range of L.

19. (8 pts) Consider the linear transformation mapping $\mathbb{R}^3$ to $\mathbb{R}^2$ given by

$$L(\boldsymbol{x}) = (x_1 - 2x_2 + x_3, x_2 + 2x_3)^T$$

Find a matrix A which represents L with respect the standard basis $[\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3]$ in $\mathbb{R}^3$ and the ordered basis $[\boldsymbol{b}_1, \boldsymbol{b}_2]$ in $\mathbb{R}^2$ where $b_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $b_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

20. (8 pts) Find the equation of the plane in $\mathbb{R}^3$ which is normal to the vector $\boldsymbol{N} = [1,\ -1,\ 2]^T$ and which passes through the point $\boldsymbol{P}_0 = [-1,\ 2,\ -5]^T$.

21. (8 pts) Find the distance in $\mathbb{R}^3$ from the point $\boldsymbol{P}_0 = [-1,\ -3,\ 1]^T$ to the plane given by $2x - 3y + z = -4$.

22. (8 pts) Let S be the subspace in $\mathbb{R}^3$ which is spanned by the set $\left\{[1,\ -2,\ 1]^T,\ [-1,\ 2,\ 2]^T\right\}$. Find a basis for $S^{\perp}$.

23. (8 pts) Consider the following orthonormal basis $U = [\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3]$ for $\mathbb{R}^3$, where

$$\boldsymbol{u}_1 = \frac{1}{\sqrt{3}}[1,\ -1,\ 1]^T,\ \boldsymbol{u}_2 = \frac{1}{\sqrt{2}}[-1,\ 0,\ 1]^T \text{ and } \boldsymbol{u}_3 = \frac{1}{\sqrt{6}}[1,\ 2,\ 1]^T.$$

Suppose L is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^4$ which satisfies $L(\boldsymbol{u}_1) = [-1,\ 0,\ 2,\ 1]^T$, $L(\boldsymbol{u}_2) = [2,\ -3,\ 0,\ -1]^T$ and $L(\boldsymbol{u}_3) = [1,\ 1,\ -1,\ -3]^T$. Find the value of $L([-1,\ 2, 1]^T)$.

24. (8 pts) Let S be the subspace of $\mathbb{R}^3$ spanned by the vectors $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ (given in Problem 23). Let $\boldsymbol{x} = [1,\ 3,\ -2]^T$. Find the projection of $\boldsymbol{x}$ onto S.

25. (10 pts) Let E be the set of linearly independent vectors $E = [\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3]$, where

$$\boldsymbol{x}_1 = [1,\ -1,\ 0,\ -1]^T,\ \boldsymbol{x}_2 = [1,\ 0,\ 0,\ -1]^T \text{ and } \boldsymbol{x}_3 = [1,\ -2,\ -1,\ 0]^T.$$

Find (using the Gram-Schmidt orthogonalization process) a set U of orthonormal vectors so that $span(E) = span(U)$.

26. (8 pts) Let $A = \begin{bmatrix} 3 & 1 & 0 \\ -6 & -2 & 0 \\ 4 & 2 & -1 \end{bmatrix}$. Find the eigenvalues of A.

27. (8 pts) Let $A = \begin{bmatrix} 1 & -2 & -2 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$. Determine whether A is diagonalizable.

Name ___________________________ Form A

Answers

1.
a. Yes No b. Yes No c. Yes No

a. Yes No b. Yes No c. Yes No

2.
a. Yes No b. Yes No c. Yes No d. Yes No

e. Yes No f. Yes No g. Yes No h. Yes No

3.
a. Yes No b. Yes No c. Yes No d. Yes No

4.
a. Yes No b. Yes No c. Yes No d. Yes No

5.
a. Yes No b. Yes No c. Yes No d. Yes No

6.
a. Yes No b. Yes No c. Yes No d. Yes No

7.
a. Yes No

8.
a. Yes No b. Yes No c. Yes No d. Yes No

9.
a. Yes No b. Yes No c. Yes No

10.
a. Yes No b. Yes No c. Yes No

11.
a. Yes No b. Yes No c. Yes No

12.
a. Yes No b. Yes No c. Yes No d. Yes No

13.
a. Yes No b. Yes No c. Yes No

d. Yes No e. Yes No f. Yes No

g. Yes No h. Yes No i. Yes No