Section 5.8

I. Integration \( \int_{a}^{b} f(x) \, dx \)

A. If possible – Apply the Fundamental Theorem of Calculus

B. If not possible – Numerical Approximation

1. Partition \([a, b]\) in \(n\) subintervals
   a. \( \Delta x = \frac{b - a}{n} \)
   b. \( x_k = a + k \Delta x \)

2. Approximation by Rectangles
   a. Right-hand Rule \( I \approx \sum_{k=1}^{n} f(x_k) \Delta x \)
   b. Mid-point Rule
      
      \[
      x_k^* = \frac{x_k + x_{k-1}}{2} = a + (k - \frac{1}{2}) \Delta x
      \]
      
      \[
      I \approx \sum_{k=1}^{n} f(x_k^*) \Delta x
      \]

3. Trapezoid Rule
   a. Area Element \( A_k = \frac{1}{2} [f(x_{k-1}) + f(x_k)] \Delta x \)
   b. \( I \approx \left[ \frac{1}{2} f(x_0) + \sum_{k=1}^{n-1} f(x_k) + \frac{1}{2} f(x_n) \right] \Delta x \)

4. Simpson’s Rule
   a. \( n \) even
   b. Area Element \( A_{2k} = \frac{1}{3} [f(x_{2k-2}) + 4f(x_{2k-1}) + f(x_{2k})] \Delta x \)
   c. \( I \approx \frac{1}{3} \left[ f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 4f(x_{n-1}) + f(x_n) \right] \Delta x \)
II. Error Estimates

A. Mid-point Rule \[ |E_n| \leq \frac{(b-a)^3}{24n^2} M \] where \( M = \max_{x \in [a,b]} |f''(x)| \)

B. Trapezoid Rule \[ |E_n| \leq \frac{(b-a)^3}{12n^2} M \] where \( M = \max_{x \in [a,b]} |f''(x)| \)

C. Simpson’s Rule \[ |E_n| \leq \frac{(b-a)^5}{180n^4} K \] where \( K = \max_{x \in [a,b]} |f^{(iv)}(x)| \)