The Invariants of the Symplectic Groups

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SUMMARY : *In this notes we study the invariant rings of the symplectic groups in odd characteristic in their tautological representation, and try to make the original paper by Carlisle and Kropholler more readable and understandable, i.e., the only new thing is the expository, in particular that/how and where the Steenrod algebra is used is my contribution.*

§1. Notation and Introduction

Let $\mathbb{F}$ be a Galois field of odd[1] characteristic p with $q = p^s$ elements. Let V be a vector space over $\mathbb{F}$ of even dimension $n = 2l$ with basis $x_1, y_1, \ldots, x_l, y_l$. Denote by

$$f : V \times V \longrightarrow \mathbb{F}$$

a non degenerate alternating bilinear form on V. Then the **symplectic group** $\mathbb{Sp}(n, \mathbb{F})$ is defined to be the group of isometries of f, or the maximal subgroup of the general linear group, $\mathrm{GL}(n, \mathbb{F})$, stabilizing[2] f

$$\begin{aligned}\mathbb{Sp}(n, \mathbb{F}) &= \mathrm{Isom}(V, f)\\ &= \{\mathbf{T} \in \mathrm{GL}(n, \mathbb{F}) \,|\, f(\mathbf{T}u, \mathbf{T}v) = f(u, v)\ \forall\ u, v \in V\}\\ &< \mathrm{GL}(n, \mathbb{F}).\end{aligned}$$

The matrix $\mathbf{A}_f$ associated to f is given by

$$(\mathbf{A}_f)_{i,j} = f(e_i, e_j),$$

for the standard basis $e_1, \ldots, e_n \in V$. After a suitable change of bases, we can assume that the matrix $\mathbf{A}_f$ has the following form

$$\mathbf{A}_f = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0\\ -1 & 0 & \cdots & 0 & 0\\ 0 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & 0 & 1\\ 0 & 0 & \cdots & -1 & 0 \end{bmatrix}.$$

So we could write the symplectic group also as

$$\mathbb{Sp}(n, \mathbb{F}) = \{\mathbf{T} \in \mathrm{GL}(n, \mathbb{F}) \,|\, \mathbf{T}^t \mathbf{A}_f \mathbf{T} = \mathbf{A}_f\},$$

i.e., as the stabilizer subgroup of $\mathbf{A}_f$ in $\mathrm{GL}(n, \mathbb{F})$.

The order of the symplectic group was calculated by L. E. Dickson:

THEOREM 1.1 (L. E. Dickson): *The order of the symplectic group in dimension $n = 2l$ over a field with $q = p^s$ elements is given by the formula*

$$|\ \mathbb{Sp}(n, \mathbb{F})\ | = q^{l^2} \prod_{i=1}^{l} (q^{2i} - 1).$$

PROOF: The proof is by counting the number of symplectic bases for V, see[3] Theorem 115 in[6] •

Thanks to Dickson we also have an exlicite set of generators for our group:

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[1] In characteristic two there are some problems in defining a group, see [12] and [4].

[2] This indeed does not depend on the choice of f since all non degenerate alternating bilinear forms are equivalent, meaning lie in the same $\mathrm{GL}(n, \mathbb{F})$-orbit. For a proof of this and an explicite introduction to groups associated to forms we refer to the fine book [12], in particular pp. 234 -246.

[3] Careful: Dickson calls the symplectic groups **special abelian groups** and denotes them by $\mathrm{SA}(n, p^s)$.

THEOREM 1.2 (L.E. Dickson): *The symplectic group* $\mathbb{Sp}(n,\ \mathbb{F})$ *is generated by the following matrices:*

$$\mathbf{S(k)} = (s_{i,j}(k)), \quad \forall\ k = 1, \ldots, l,$$

where

$$s_{i,j}(k) = \begin{cases} 1 & \text{if } i = j \neq k \\ 1 & \text{if } i = 2k-1,\ j = 2k \\ -1 & \text{if } i = 2k,\ j = 2k-1 \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathbf{S(k,}\ \lambda) = (s_{i,j}(k,\ \lambda)), \quad \forall\ k = 1, \ldots, l \ \text{ and } \lambda \in \mathbb{F}^{\times},$$

where

$$s_{i,j}(k,\ \lambda) = \begin{cases} 1 & \text{if } i = j \\ \lambda & \text{if } i = 2k-1,\ j = 2k \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\mathbf{S(k,\ l,}\ \lambda) = (s_{i,j}(k,\ l,\ \lambda)), \quad \forall\ k,\ l = 1, \ldots, l \ \text{ and } \lambda \in \mathbb{F}^{\times},$$

where

$$s_{i,j}(k,\ l,\ \lambda) = \begin{cases} 1 & \text{if } i = j \\ \lambda & \text{if } i = 2k-1,\ j = 2l \\ \lambda & \text{if } i = 2l-1,\ j = 2k \\ 0 & \text{otherwise.} \end{cases}$$

PROOF: See Theorem 114 in [6] •

As Dickson remarks, we can read off this explicite description that the symplectic group is generated by symplectic transvections, and therefore every element has determinant one. This will turn out to be of great use later.

During this manuscript we will have to struggle with a lot of awful formulae. Therefore I have explicitely calculated most of them for the case $n = 4$ (well, $n = 2$ isn't really[4] exciting). So, if you feel overwhelmed by indices, you are welcome to consult the final Section 7.

§2. A Bunch of Invariants

We are going to find a collection of polynomial invariants of our symplectic group, which will, together with the Dickson classes and the Euler class, lead to a complete set of algebra generators of the ring of invariants.

PROPOSITION 2.1: *For any natural number* $i \in \mathbb{N}$

$$\xi_{n,\ i} := \sum_{j=1}^{l} (x_j y_j^{q^i} - y_j x_j^{q^i}) \in \mathbb{F}[x_1,\ y_1, \ldots,\ x_l,\ y_l]^{\mathbb{Sp}(n,\ \mathbb{F})}.$$

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[4] In dimension two the symplectic group $\mathbb{Sp}(2,\ \mathbb{F})$ is nothing but the special linear group $\mathrm{SL}(2,\ \mathbb{F})$. To see this you could use Dickson's theorem to find that the symplectic group is contained in the special linear, because all elements have determinant one, and then use again Dickson to find that they have the same order. On the other hand you could replace the first step by show by ordinary explicite matrix calculations that the special linear matrices are symplectic.

PROOF: Observe that

$$\xi_n := x_1 \wedge y_1 + \cdots + x_l \wedge y_l \in \left(\Lambda^2(V^*)\right)^{\mathbb{Sp}(n,\ \mathbb{F})} \subseteq \left(\mathbb{F}[V] \otimes_{\mathbb{F}} \Lambda^*(V^*)\right)^{\mathbb{Sp}(n,\ \mathbb{F})}$$

is a non degenerate alternating bilinear form and hence, by definition of the symplectic group, invariant under the action of $\mathbb{Sp}(n,\ \mathbb{F})$. Since

$$\begin{aligned} \xi_{n,1} &:= \sum_{j=1}^{l} (x_j y_j^q - y_j x_j^q) \\ &= \beta \mathcal{P}^1 \beta(\xi_n), \end{aligned}$$

where β denotes the Bockstein operator

$$\begin{array}{ccc} \beta : \mathbb{F}[V] \otimes_{\mathbb{F}} \Lambda^*(V^*) & \longrightarrow & \mathbb{F}[V] \otimes_{\mathbb{F}} \Lambda^*(V^*) \\ (z_i,\ 0) & \longmapsto & (0,\ 0) \\ (0,\ z_i) & \longmapsto & (z_i,\ 0) \\ (0,\ z_{i_1} \wedge \ldots \wedge z_{i_j}) & \longmapsto & \sum_{k=1}^{j} (-1)^{k-1} (z_{i_k},\ z_{i_1} \wedge \ldots \wedge \widehat{z_{i_k}} \wedge \ldots \wedge z_{i_j}), \end{array}$$

for $z_i \in \{x_1,\ y_1, \ldots,\ x_l,\ y_l\}$, and $\mathcal{P}^1$ the first Steenrod reduced power operation, this is also invariant, hence

$$\xi_{n,1} \in \mathbb{F}[V]^{\mathbb{Sp}(n,\ \mathbb{F})}.$$

Moreover we have[5]

$$\xi_{n,i+1} = \mathcal{P}^{q^i}(\xi_{n,i}),$$

and therefore our statement is proved •

Here is another gorgeous observation by Carlisle and Kropholler:

LEMMA 2.2: *The symplectic group is precisely the stabilizer subgroup of* $\xi_{n,1}$ *in the general linear group*

$$\mathbb{Sp}(n,\ \mathbb{F}) = \mathrm{GL}(n,\ \mathbb{F})_{\xi_{n,1}}.$$

PROOF: By the preceding lemma we know that $\xi_{n,1}$ is an invariant and therefore

$$\mathbb{Sp}(n,\ \mathbb{F}) \hookrightarrow \mathrm{GL}(n,\ \mathbb{F})_{\xi_{n,1}}.$$

On the other hand the map[6]

$$\beta \mathcal{P}^1 \beta : \Lambda^2(V^*) \hookrightarrow \mathbb{F}[V]_{(q+1)}$$

is injective. Therefore for any $g \in \mathrm{GL}(n,\ \mathbb{F})_{\xi_{n,1}}$, i.e., $g\xi_{n,1} = \xi_{n,1}$, also

$$g\xi_n = \xi_n.$$

This in turn means that $g \in \mathbb{Sp}(n,\ \mathbb{F})$, since the symplectic group is defined to be the stabilizer of ξ_n •

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[5] If you can't do this calculation by yourself, you might look it up (Yes! There is a back of the book!) in Lemma 4.1, where all Steenrod powers of the ξ's are calculated.

[6] For a graded algebra A we denote the homogeneous part of degree d by $A_{(d)}$. By the way, Carlisle und Kropholler call this map *symmetrizing map*.

REMARK: In the same way we could observe that also the maps

$$\beta\mathcal{P}^{q^i}\mathcal{P}^{q^{i-1}}\cdots\mathcal{P}^1\beta : \Lambda^2(V^*) \hookrightarrow \mathbb{F}[V]_{(q^{i+1}+1)}$$

are injective, sending ξ_n to $\xi_{n,i+1}$. So that the symplectic group is also the stabelizer of all the other $\xi_{n,i+1}$'s

$$\mathbb{S}\mathbb{p}(n,\ \mathbb{F}) = \mathrm{GL}(n,\ \mathbb{F})_{\xi_{n,i+1}} \quad \forall\ i,$$

what in turn could replace the preceding Proposition 2.1.

Next we show that the maximal possible number of the invariants just constructed, namely n of them, are algebraically independent.

LEMMA 2.3: *The polynomials*

$$\xi_{n,1}, \ldots, \xi_{n,n} \in \mathbb{F}[V]^{\mathbb{S}\mathbb{p}(n,\ \mathbb{F})}$$

are algebraically independent.

PROOF: We show this by calculating the determinant of the Jacobian matrix, $Jac(\xi_{n,1}, \ldots, \xi_{n,n})$, associated to these polynomials. To this end note that for all $j = 1, \ldots, l$ and $i \in \mathbb{N}$

$$\frac{\partial \xi_{n,i}}{\partial x_j} = y_j^{q^i}$$

and

$$\frac{\partial \xi_{n,i}}{\partial y_j} = -x_j^{q^i}.$$

Therefore we get

$$\det\big(Jac(\xi_{n,1}, \ldots, \xi_{n,n})\big) = \det\begin{bmatrix} y_1^q & -x_1^q & \cdots & y_l^q & -x_l^q \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ y_1^{q^n} & -x_1^{q^n} & \cdots & y_l^{q^n} & -x_l^{q^n} \end{bmatrix}.$$

However, the latter is, up to sign, the q-th power of the Euler class $\mathbf{E}_n$ (of the orbit $V^* \setminus 0$ of $\mathrm{SL}(n,\ \mathbb{F})$ on V^*) and in particular non zero. That's all we need •

We note the obvious corollary:

COROLLARY 2.4:

$$\mathbb{F}[\xi_{n,1}, \ldots, \xi_{n,n}] \hookrightarrow \mathbb{F}[V]^{\mathbb{S}\mathbb{p}(n,\ \mathbb{F})}$$

is a polynomial subalgebra.

Lets remark at this point also that a symplectic matrix has determinant one, because the symplectic group is generated by symplectic transvections, see Theorem 114 in [6] or, for a more modern treatment, e.g., [8] Lemma 1 on Section 6.9, and therefore

$$\mathbb{F}[\mathbf{E}_n,\ \mathbf{d}_{n,\ 1},\ , \ldots, \mathbf{d}_{n,\ n-1}] = \mathbb{F}[V]^{\mathrm{SL}(n,\ \mathbb{F})} \hookrightarrow \mathbb{F}[V]^{\mathbb{S}\mathbb{p}(n,\ \mathbb{F})},$$

where $\mathbf{E}_n$ denotes the Euler class of the orbit $V^* \setminus 0$ of $\mathrm{SL}(n,\ \mathbb{F})$ on V^*.

In a later section it will turn out that these invariants, namely

$$\xi_{n,1}, \ldots, \xi_{n,n},\ \mathbf{E}_n,\ \mathbf{d}_{n,\ 1}, \ldots, \mathbf{d}_{n,\ n-1}$$

form a complete set of algebra generators[7] of the ring of invariants $\mathbb{F}[V]^{\mathbb{Sp}(n,\,\mathbb{F})}$, in other words, we will show that

$$A := \mathbb{F} < \xi_{n,1}, \dots, \xi_{n,n}, \ \mathbf{E}_n, \ \mathbf{d}_{n,\,1}, \dots, \mathbf{d}_{n,\,n-1} >,$$

where $\mathbb{F} < - >$ denotes the $\mathbb{F}$-algebra generated by the stuff in the $< - >$-brackets (which is possibly *not* a polynomial algebra), is precisely the ring of invariants.

We do this by determine also the relations among these generators, which leads to a dozen pages of calculations. Then we still have to *prove* that that's it: that takes another 6 pages, i.e., it's probably not really pleasant to read all this (and it is certainly not pleasant to give a talk about it).

§3. A Bunch of Relations

In this section we calculate some relations between the invariants we have so far.

We need some preliminaries.

Define an alternating bilinear form on the n-fold direct product $\times_n \mathbb{F}[V]$ by (n=2l remember)

$$< F, \ H >:= \sum_{j=1}^{l} (f_{2j-1} h_{2j} - f_{2j} h_{2j-1}),$$

for n-tuples $F = (f_1, \dots, f_{2l})$ and $H = (h_1, \dots, h_{2l})$. Denote by

$$\tau := (12)(34)(\cdots)(2l-1 \ 2l) \in \Sigma_{2l}$$

and take its centralizer in the symmetric group

$$C(\tau) < \Sigma_{2l}.$$

Next define

$$|| F^{(1)}, \dots, F^{(2l)} || := \sum_{\sigma} \left(\operatorname{sign}(\sigma) \prod_{j=1}^{l} < F^{\sigma(2j-1)}, \ F^{\sigma(2j)} > \right),$$

where the sum runs over a set of coset representatives of $C(\tau)$ in Σ_{2l}.

LEMMA 3.1: *With the preceding notation we have*

(1) $|| \cdots ||$ *is independent of the choice of the coset representatives.*
(2) $|| \cdots ||$ *is multilinear and alternating.*
(3) $|| e^{(1)}, \dots, e^{(2l)} || = 1$ *where* $e^{(i)}$ *denotes the* i*-th standard basis vector* $(0, \dots, 0, 1, 0, \dots, 0)$.

PROOF:

AD (1) : The centralizer of the element $\tau = (12)(34)\cdots(2l-1 \ 2l) \in \Sigma_{2l}$ can be expressed in the following way: Take an embedding

$$\Sigma_l \hookrightarrow \Sigma_{2l}, \ \sigma \longmapsto \bar{\sigma},$$

where we define the image for any $k = 1, \dots, l$ by

$$\bar{\sigma}(2k-1) := 2\sigma(k) - 1$$
$$\bar{\sigma}(2k) := 2\sigma(k).$$

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[7] Note that these generators arise in a very natural way: the Dickson classes, an Euler class, the form ξ_n that defines the group and its Steenrod powers.

Then

$$\begin{aligned} C(\tau) &= \langle (12),\ \Sigma_l \rangle \subset \Sigma_{2l} \\ &= \underset{\longleftarrow\, l \,\longrightarrow}{\mathbb{Z}/2 \times \cdots \times \mathbb{Z}/2} \rtimes \Sigma_l \\ &= \mathbb{Z}/2 \wr \Sigma_l, \end{aligned}$$

so the Σ_l permutes the l transpositions $(12), \ldots, (2l-1\ 2l)$ of τ and the $\mathbb{Z}/2$ permutes the entries $2k-1,\ 2k,\ \forall\ k$ of the transpositions. That this is indeed the full centralizer is an easy calculation. An equally straightforward calculation shows that our sum is the same if we replace σ by $\sigma\pi$ where π is one of the generators of $C(\tau)$, because we just permute the order of the product or the order of the pairs in the factors.

AD (2) : Since $< -,\ - >$ is bilinear, we have that $|| \cdots ||$ is multilinear. To prove that the form is alternating assume that $F^{(j_0)} = F^{(k_0)} = F$ for some $k_0,\ j_0 = 1, \ldots, n$. If

$$\sigma_1(2i_0 - 1) = j_0 \quad \text{and} \quad \sigma_1(2i_0) = k_0$$

(or vice versa) for some $i_0 = 1, \ldots, l$ and some σ_1, then

$$< F^{\sigma_1(2i_0-1)},\ F^{\sigma_1(2i_0)} > = < F,\ F > = 0,$$

i.e., the product $\prod_j < F^{\sigma_1(2j-1)},\ F^{\sigma_1(2j)} > = < F,\ F > \prod_{j \neq i_0} < F^{\sigma_1(2j-1)},\ F^{\sigma_1(2j)} >$ vanishes. If

$$\sigma_2(2i_0 - 1) = j_0$$

and

$$\sigma_2(2i_0' - 1) = k_0 \quad \left(\text{or} \quad \sigma_2(2i_0') = k_0\right)$$

(or vice versa) for some $i_0,\ i_0' = 1, \ldots, l$ and some σ_2, then

$$\prod_j < F^{\sigma_2(2j-1)},\ F^{\sigma_2(2j)} > = < F,\ F^{\sigma_2(2i_0)} > < F,\ F^{\sigma_2(2i_0')} > \prod_{j \neq i_0,\ i_0'} < F^{\sigma_2(2j-1)},\ F^{\sigma_2(2j)} > .$$

Consider the element $\sigma_2(2i_0\ 2i_0') \in \Sigma_{2l}/C(\tau)$. It gives the same summand as σ_2, namely

$$\begin{aligned} &\prod_j < F^{\left(\sigma_2(2i_0\ 2i_0')\right)(2j-1)},\ F^{\left(\sigma_2(2i_0\ 2i_0')\right)(2j)} > \\ &\qquad = < F,\ F^{\sigma_2(2i_0')} > < F,\ F^{\sigma_2(2i_0)} > \prod_{j \neq i_0,\ i_0'} < F^{\left(\sigma_2(2i_0\ 2i_0')\right)(2j-1)},\ F^{\left(\sigma_2(2i_0\ 2i_0')\right)(2j)} >, \end{aligned}$$

but with the opposite sign, because $\operatorname{sign}(\sigma_2) = -\operatorname{sign}(\sigma_2(2i_0\ 2i_0'))$. So the two summands cancel. Altogether we have

$$\begin{aligned} ||F^{(1)}, \ldots, F^{(2l)}|| &= \sum_\sigma \left(\operatorname{sign}(\sigma) \prod_{j=1}^{l} < F^{\sigma(2j-1)},\ F^{\sigma(2j)} > \right) \\ &= \sum_{\sigma_1} \left(\operatorname{sign}(\sigma_1) < F,\ F > \prod_{j \neq i_0} < F^{\sigma_1(2j-1)},\ F^{\sigma_1(2j)} > \right) \\ &\quad + \sum_{\sigma_2} \Big(\operatorname{sign}(\sigma_2) < F,\ F^{\sigma_2(2i_0)} >< F,\ F^{\sigma_2(2i_0')} > \end{aligned}$$

$$\left(\prod_{j \neq i_0,\ i_0'} < F^{\left(\sigma_2(2i_0\ 2i_0')\right)(2j-1)},\ F^{\left(\sigma_2(2i_0\ 2i_0')\right)(2j)} > \right.$$

$$\left. - \prod_{j \neq i_0,\ i_0'} < F^{\sigma_2(2j-1)},\ F^{\sigma_2(2j)} > \right)$$

$$= 0.$$

AD (3) : We have

$$< e^{(i)},\ e^{(j)} >= \begin{cases} +1 & \text{if } i \text{ is odd and } j = i + 1 \\ -1 & \text{if } i \text{ is even and } j = i - 1 \\ 0 & \text{otherwise.} \end{cases}$$

Hence all summands vanish except the one where σ = id

$$||e^{(1)}, \ldots,\ e^{(2l)}|| = \sum_{\sigma} \left(\operatorname{sign}(\sigma) \prod_{j=1}^{l} < e^{\sigma(2j-1)},\ e^{\sigma(2j)} > \right)$$

$$= \prod_{j=1}^{l} < e^{(2j-1)},\ e^{(2j)} >$$

$$= 1.$$

That's all we claimed •

So, the first statement of the lemma tells us that our definition of $|| \cdots ||$ is welldefined, while the second and third imply that

$$||F^{(1)}, \ldots,\ F^{(2l)}|| = \det \begin{bmatrix} F^{(1)} \\ \vdots \\ F^{(2l)} \end{bmatrix}.$$

We are now prepared to prove the following lemma.

PROPOSITION 3.2: *With the notation above we have:*

1. *The Euler class* $\mathbf{E}_n \in \mathbb{F}[\xi_{n,1}, \ldots,\ \xi_{n,n-1}]$ *is irreducible in* $\mathbb{F}[V]^{\mathbb{Sp}(n,\ \mathbb{F})}$.
2. $\mathbf{E}_n \mathbf{d}_{n,\ i} \in \mathbb{F}[\xi_{n,1}, \ldots,\ \xi_{n,n}]$ *for all* $i = 1, \ldots,\ n-1$.
3. *If for some polynomial* $f \in \mathbb{F}[V]$ *we have that* $\mathbf{E}_n f \in \mathbb{F}[\xi_{n,1}, \ldots,\ \xi_{n,n-1}]$, *then* $f \in \mathbb{F}[\xi_{n,1}, \ldots,\ \xi_{n,n-1}]$.

PROOF: We take the things in order.

AD 1 : Define $2l$-tuples $F^{(j)} := (x_1^{q^{j-1}},\ y_1^{q^{j-1}}, \ldots,\ x_l^{q^{j-1}},\ y_l^{q^{j-1}}) \in \underset{2l}{\times} \mathbb{F}[V]$ of polynomials for $j \in \mathbb{N}$. Then together with the preceding observation the Euler class $\mathbf{E}_n$ is, according to Dickson [6], given by

$$\mathbf{E}_n = \det \begin{bmatrix} x_1 & y_1 & \cdots & x_l & y_l \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ x_1^{q^{2l-1}} & y_1^{q^{2l-1}} & \cdots & x_l^{q^{2l-1}} & y_l^{q^{2l-1}} \end{bmatrix}$$

$$= \det \begin{bmatrix} F^{(1)} \\ \vdots \\ F^{(2l)} \end{bmatrix}$$

$$= ||F^{(1)}, \ldots,\ F^{(2l)}||$$

$$= \sum_{\sigma} \left(\operatorname{sign}(\sigma) \prod_{j=1}^{l} < F^{\sigma(2j-1)},\ F^{\sigma(2j)} > \right).$$

Since

$$< F^{(i)},\ F^{(j)} >= \xi_{n,|j-i|}^{q^{\min(i,\ j)-1}}$$

as one easily calculates, and for our choice $|j-i| \leq 2l-1$ we get the desired inclusion

$$\mathbf{E}_n \in \mathbb{F}[\xi_{n,1}, \ldots, \xi_{n,n-1}],$$

i.e., there exists a polynomial $\mathcal{E} = \mathcal{E}(X_1, \ldots, x_{n-1}) \in \mathbb{F}[X_1, \ldots, X_{n-1}]$ in $n-1$ indeterminants such that

$$\mathbf{E}_n = \mathcal{E}(\xi_{n,1}, \ldots, \xi_{n,n-1}).$$

The Euler class is by construction the product of linear forms where we take for each one dimensional vector subspace of V^* exactly one form. Moreover the symplectic group $\mathbb{Sp}(n,\ \mathbb{F})$ acts transitively[8] on the set of hyper planes of V. Hence $\mathbf{E}_n$ is irreducible in $\mathbb{F}[V]^{\mathbb{Sp}(n,\ \mathbb{F})}$ and a fortiori in $\mathbb{F}[\xi_{n,1}, \ldots, \xi_{n,n-1}]$.

AD 2 : Since (again thanks to Dickson)

$$\mathbf{E}_n \mathbf{d}_{n,\ i} = \det \begin{bmatrix} x_1^{q^{i_1}} & y_1^{q^{i_1}} & \cdots & x_l^{q^{i_1}} & y_l^{q^{i_1}} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ x_1^{q^{in}} & y_1^{q^{in}} & \cdots & x_l^{q^{in}} & y_l^{q^{in}} \end{bmatrix}$$

for $0 \leq i_1 < i_2 < \cdots < i_{n-1} < i_n \leq n$ where $i_j \neq i$ we can proceed as in (1) and get

$$\mathbf{E}_n \mathbf{d}_{n,\ i} = \sum_{\sigma} \left(\operatorname{sign}(\sigma) \prod_{j=1}^{l} < F^{\sigma(2j-1)},\ F^{\sigma(2j)} > \right) \in \mathbb{F}[\xi_{n,1}, \ldots, \xi_{n,n}].$$

In other words there exist polynomials in n indeterminants

$$\mathcal{D}_{n,i} = \mathcal{D}_{n,i}(X_1, \ldots, X_n) \in \mathbb{F}[X_1, \ldots, X_n]$$

such that

$$\mathcal{D}_{n,i}(\xi_{n,1}, \ldots, \xi_{n,n}) = \mathbf{E}_n \mathbf{d}_{n,\ i},$$

for all $i = 0, \ldots, n-1$.

AD 3 : Consider the composition of maps

$$\varphi : \mathbb{F}[\xi_{n,1}, \ldots, \xi_{n,n-1}] \hookrightarrow \mathbb{F}[V] \longrightarrow \mathbb{F}[x_1,\ y_1, \ldots, x_{l-1},\ y_{l-1}],$$

where the first is just the canonical inclusion of algebras while the second is induced by the inclusion of vector spaces

$$\operatorname{Span}_{\mathbb{F}}(e_1, \ldots, e_{2l-2}) \hookrightarrow V,$$

where $e_1, \ldots, e_{2l-2}$ denotes the dual basis to $x_1,\ y_1, \ldots, x_{l-1},\ y_{l-1}$. By Lemma 2.3 the polynomials $\varphi(\xi_{n,1}) = \xi_{n-2,1}, \ldots, \varphi(\xi_{n,n-2}) = \xi_{n-2,n-2}$ are algebraically independent. Therefore the image of φ generates a subalgebra of Krull dimension at least $n-2$. Since $n-2 = \dim(\mathbb{F}[x_1, \ldots, y_{l-1}]$ the kernel of φ is a prime ideal of height 1. Obviously the Euler class $\mathbf{E}_n$ is in the kernel. By (1) the class $\mathbf{E}_n$ is irreducible, and therefore prime, because $\mathbb{F}[\xi_{n,1}, \ldots, \xi_{n,n-1}]$ is a unique factorization domain, i.e.,

$$(\mathbf{E}_n) = \ker(\varphi) \subset \mathbb{F}[\xi_{n,1}, \ldots, \xi_{n,n-1}].$$

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[8] This is just an application of Witt's Lemma, see, e.g., Section 20 in [1], or Lemma 3 in Section 6.9 of [8].

Denote by $< \mathbf{E}_n > \subset \mathbb{F}[V]$ the principle ideal generated by the Euler class in the full polynomial ring $\mathbb{F}[V]$. Certainly we have

$$(\mathbf{E}_n) \subseteq < \mathbf{E}_n > \cap\, \mathbb{F}[\xi_{n,1}, \ldots, \xi_{n,n-1}].$$

Since for every element in the big ideal $f\mathbf{E}_n \in < \mathbf{E}_n > \cap\, \mathbb{F}[\xi_{n,1}, \ldots, \xi_{n,n-1}]$ *(i.e., for all* $f \in \mathbb{F}[V]$*)* we have that $\varphi(f\mathbf{E}_n) = 0$, the two ideals in question must be equal, i.e., the polynomial $f \in \mathbb{F}[\xi_{n,1}, \ldots, \xi_{n,n-1}]$, as we wanted to show •

We need to be more precise about the polynomials $\mathcal{D}_{n,i}$ occuring in part (2).

LEMMA 3.3 : *For* $i = 0, \ldots, n-1$ *the polynomials* $\mathcal{D}_{n,i}(X_1, \ldots, X_n)$ *are linear in* X_n *with leading coefficient*

$$\mathcal{D}_{n-2,i-1}(X_1, \ldots, X_n)^q.$$

PROOF: Recall that

$$\begin{aligned}
\mathbf{E}_n\mathbf{d}_{n,\,i} = \ &\det \begin{bmatrix} x_1^{q^{i_1}} & y_1^{q^{i_1}} & \cdots & x_l^{q^{i_1}} & y_l^{q^{i_1}} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ x_1^{q^{i_n}} & y_1^{q^{i_n}} & \cdots & x_l^{q^{i_n}} & y_l^{q^{i_n}} \end{bmatrix} \\
= \ &\det \begin{bmatrix} F^{(1)} \\ \vdots \\ F^{(2l)} \end{bmatrix} \\
= \ &||F^{(1)}, \ldots, F^{(2l)}|| \\
= \ &\sum_{\sigma} \left(\operatorname{sign}(\sigma) \prod_{j=1}^{l} < F^{\sigma(2j-1)},\ F^{\sigma(2j)} > \right) \\
= \ &\sum_{\{\sigma, |\sigma(2j-1) - \sigma(2j) - 1| \neq n, \forall j\}} \left(\operatorname{sign}(\sigma) \prod_{j=1}^{l} < F^{\sigma(2j-1)},\ F^{\sigma(2j)} > \right) \\
&+ \sum_{\{\sigma, |\sigma(2j_0-1) - \sigma(2j_0) - 1| = n\}} \left(\operatorname{sign}(\sigma) \prod_{j=1, \neq j_0}^{l} < F^{\sigma(2j-1)},\ F^{\sigma(2j)} > \xi_{n,n} \right) \\
= \ &\sum_{\{\sigma, |\sigma(2j-1) - \sigma(2j) - 1| \neq n, \forall j\}} \left(\operatorname{sign}(\sigma) \prod_{j=1}^{l} < F^{\sigma(2j-1)},\ F^{\sigma(2j)} > \right) \\
&+ \left(\sum_{\{\sigma, |\sigma(2j_0-1) - \sigma(2j_0) - 1| = n\}} \left(\operatorname{sign}(\sigma) \prod_{j=1, \neq j_0}^{l} < F^{\sigma(2j-1)},\ F^{\sigma(2j)} > \right) \right) \xi_{n,n}.
\end{aligned}$$

Therefore $\mathbf{E}_n\mathbf{d}_{n,\,i}$ is linear in $\xi_{n,\,n}$. To find the leading coefficient we have to work a bit harder and calculate the sum

$$\sum_{\{\sigma, |\sigma(2j_0-1) - \sigma(2j_0) - 1| = n\}} \left(\operatorname{sign}(\sigma) \prod_{j=1, \neq j_0}^{l} < F^{\sigma(2j-1)},\ F^{\sigma(2j)} > \right).$$

Note first of all that we can assume without loss of generality that $j_0 = l$ (if not we replace σ by $\sigma\gamma$

where $\gamma \in C(\tau)$ interchanges j_0 and l). So we have to calculate

$$\sum_{\{\sigma, |\sigma(2l-1)-\sigma(2l)-1|=n\}} \left(\operatorname{sign}(\sigma)\prod_{j=1}^{l-1} < F^{\sigma(2j-1)},\ F^{\sigma(2j)} >\right),$$

where the exponents $\sigma(2j-1),\ \sigma(2j) \in \{2, \ldots, n-1\}$. That means that the involved vectors F *all* are q-th powers (because $F^{(1)}$ does not occur anymore) of, say, $\tilde{F}$, i.e., we have

$$\left(\sum_{\{\sigma, |\sigma(2l-1)-\sigma(2l)-1|=n\}} \left(\operatorname{sign}(\sigma)\prod_{j=1}^{l-1} < \tilde{F}^{\sigma(2j-1)},\ \tilde{F}^{\sigma(2j)} >\right)\right)^{q}.$$

Note that we are still summing over coset representatives σ of the centralizer $C(\tau)$ in Σ_{2l}, with the only restriction that our σ's look[9] like $(n-1\ 1)(n)\tilde{\sigma}$, where $\tilde{\sigma} \in \Sigma_{2l-2}$. We have to convince ourselves that the elements $\tilde{\sigma}$ run over a complete set of coset representatives (exactly once) of $C(\tau_{2l-2})$ in Σ_{2l-2}, where we set $\tau_{2l-2} := (12)\cdots(2l-3\ 2l-2) \in \Sigma_{2l-2}$. Define a map

$$\Sigma_{2l-2}/C(\tau_{2l-2}) \longrightarrow \left\{(n-1\ 1)(n)\tilde{\sigma}\, C(\tau_{2l})\right\},\ \tilde{\sigma}\, C(\tau_{2l-2}) \longmapsto (n-1\ 1)(n)\tilde{\sigma}\, C(\tau_{2l}).$$

This map is obviously injective. We define a splitting via

$$\left\{(n-1\ 1)(n)\tilde{\sigma}\, C(\tau_{2l})\right\} \longrightarrow \Sigma_{2l-2}/C(\tau_{2l-2}),\ (n-1\ 1)(n)\tilde{\sigma}\, C(\tau_{2l}) \longmapsto \tilde{\sigma}\, C(\tau_{2l-2}).$$

This map is equally injective, because if we take two different elements

$$(n-1\ 1)(n)\tilde{\sigma}_1 C(\tau_{2l}) \neq (n-1\ 1)(n)\tilde{\sigma}_2 C(\tau_{2l}),$$

then also $\tilde{\sigma}_1 C(\tau_{2l-2}) \neq \tilde{\sigma}_2 C(\tau_{2l-2})$.

Coming back to our coefficient we summarize

$$\begin{aligned}
\mathbf{E}_n \mathbf{d}_{n,\ i} = & \sum_{\{\sigma, |\sigma(2j-1)-\sigma(2j)-1| \neq n, \forall j\}} \left(\operatorname{sign}(\sigma)\prod_{j=1}^{l} < F^{\sigma(2j-1)},\ F^{\sigma(2j)} >\right) \\
& + \left(\sum_{\{\sigma, |\sigma(2j_0-1)-\sigma(2j_0)-1|=n\}} \left(\operatorname{sign}(\sigma)\prod_{j=1, \neq j_0}^{l} < F^{\sigma(2j-1)},\ F^{\sigma(2j)} >\right)\right) \xi_{n,n} \\
= & \sum_{\{\sigma, |\sigma(2j-1)-\sigma(2j)-1| \neq n, \forall j\}} \left(\operatorname{sign}(\sigma)\prod_{j=1}^{l} < F^{\sigma(2j-1)},\ F^{\sigma(2j)} >\right) \\
& + \left(\sum_{\{\sigma, |\sigma(2l-1)-\sigma(2l)-1|=n\}} \operatorname{sign}(\sigma)\prod_{j=1}^{l-1} \left(< \tilde{F}^{\sigma(2j-1)},\ \tilde{F}^{\sigma(2j)} >\right)^{q}\right) \xi_{n,\ n} \\
= & \sum_{\{\sigma, |\sigma(2j-1)-\sigma(2j)-1| \neq n, \forall j\}} \left(\operatorname{sign}(\sigma)\prod_{j=1}^{l} < F^{\sigma(2j-1)},\ F^{\sigma(2j)} >\right) \\
& + \left(\sum_{\tilde{\sigma} \in C(\tau_{2l-2})} \left(\operatorname{sign}(\tilde{\sigma})\prod_{j=1}^{l-1} < \tilde{F}^{\tilde{\sigma}(2j-1)},\ \tilde{F}^{\tilde{\sigma}(2j)} >\right)\right)^{q} \xi_{n,\ n}
\end{aligned}$$

---

[9] The (n) emphasizes that the σ's fix n.

$$= \sum_{\{\sigma, |\sigma(2j-1)-\sigma(2j)-1| \neq n, \forall j\}} \left(\operatorname{sign}(\sigma) \prod_{j=1}^{l} < F^{\sigma(2j-1)},\ F^{\sigma(2j)} > \right)$$
$$+ \ \left(\mathcal{D}_{n-2,\ i-1}(\xi_{n,\ 1}, \ldots, \ \xi_{n,\ n-2})^q \right) \xi_{n,\ n},$$

where in the last step we used that

$$\tilde{F}^j = (x_1^{q^{i_j}-1}, \ldots, \ y_l^{q^{i_j}-1}) \quad \forall\ j$$

by construction. Since the invariants $\xi_{n,1}, \ldots, \ \xi_{n,n}$ are algebraically independent by Lemma 2.3 this proves, for all $i = 0, \ldots, \ n-1$,

$$\mathcal{D}_{n,i}(X_1, \ldots, \ X_n) = \left(\mathcal{D}_{n-1,i-1}(X_1, \ldots, \ X_{n-2})\right)^q X_n + \text{junk},$$

where junk does not depend on X_n •

A similar construction leads to another relation. For that we need the following lemma.

LEMMA 3.4: *Let $F^{(1)}, \ldots, \ F^{(n+2)} \in \underset{n}{\times}\mathbb{F}[V]$ be $n+2$ n-tuples of polynomials. Then*

$$\wr\wr F^{(1)}, \ldots, \ F^{(n+2)} \wr\wr := \sum_{k=2}^{n+2} (-1)^k < F^{(1)},\ F^{(k)} > ||F^{(2)}, \ldots, \ \widehat{F^{(k)}}, \ldots, \ F^{(n+2)}||$$

defines an alternating multilinear form.

PROOF: Turn the F's into $(n+2)$-tuples of polynomials by adding two zero entries at the end and note that this does not change the value of $< F^j,\ F^k >$ for any $j,\ k$. We have seen in Lemma 3.1 that

$$||F^{(1)}, \ldots, \ F^{(n+2)}|| := \sum_{\sigma} \left(\operatorname{sign}(\sigma) \prod_{j=1}^{l+1} < F^{\sigma(2j-1)},\ F^{\sigma(2j)} > \right)$$

is a multilinear alternating form. In every product the factor $< F^{(1)},\ F^{(k)} >$ occurs for some k, i.e.,

$$\sigma_k(2j_0 - 1) = 1 \quad \text{and} \quad \sigma_k(2j_0) = k$$

(or vice versa) for some $j_0 = 1, \ldots, \ l$. Without loss of generality we can assume that $\sigma_k(1) = 1$ and $\sigma_k(2) = k$ for otherwise we replace σ_k by $\sigma_k(2j_0 - 1\ 1)(2j_0\ 2)$, which represents the same coset. Hence we have

$$\wr\wr F^{(1)}, \ldots, \ F^{(n+2)} \wr\wr = \sum_{\sigma} \operatorname{sign}(\sigma) < F^{(1)},\ F^{(k)} > \left(\prod_{j=2}^{l+1} < F^{\sigma(2j-1)},\ F^{\sigma(2j)} > \right).$$

Like in the preceding lemma we replace σ_k by $\tilde{\sigma}$ where $\tilde{\sigma} \in \Sigma_{2l}$ permutes the set $\{3, \ldots, \ 2l+2\}$ and we observe that the permutations $\tilde{\sigma}$ run through a complete set of coset representatives of $C(\tau_{2l})$ in Σ_{2l}. Moreover observe that $\operatorname{sign}(\sigma) = (-1)^k \operatorname{sign}(\tilde{\sigma})$, because the number of descents of σ is precisely k plus the number of descents of $\tilde{\sigma}$. Therefore we have

$$\wr\wr F^{(1)}, \ldots, \ F^{(n+2)} \wr\wr$$
$$= \sum_{\sigma} \operatorname{sign}(\sigma) < F^{(1)},\ F^{(k)} > \prod_{j=2}^{l+1} < F^{\sigma(2j-1)},\ F^{\sigma(2j)} >$$
$$= \sum_{k=2}^{n+2} (-1)^k < F^{(1)},\ F^{(k)} > \left(\sum_{\tilde{\sigma}} \operatorname{sign}(\tilde{\sigma}) \prod_{j=2}^{l+1} < F^{\tilde{\sigma}(2j-1)},\ F^{\tilde{\sigma}(2j)} > \right)$$
$$= \sum_{k=2}^{n+2} (-1)^k < F^{(1)},\ F^{(k)} > ||F^{(2)}, \ldots, \ \widehat{F^{(k)}}, \ldots, \ F^{(n+2)}||$$

as claimed •

CONVENTION: Denote $\mathbf{d}_{n,n} = 1$ and $\mathbf{d}_{n,j} = 0$, whenever $j \notin \{0, \dots, n\}$. Then setting $\mathcal{D}_{n,\ n} = \mathcal{E}_n$ makes the whole story consistent.

PROPOSITION 3.5: *We have*

$$P_0 := \sum_{j=1}^{n} (-1)^j \xi_{n,j} \mathbf{d}_{n,j} = 0$$

in $\mathbb{F}[V]^{\mathbb{Sp}(n,\ \mathbb{F})}$.

PROOF: We set

$$F = F^{(1)} = F^{(2)} = (x_1,\ y_1, \dots,\ x_l,\ y_l)$$

and for $j = 3, \dots, n+2$

$$F^{(j)} = (x_1^{q^{j-2}},\ y_1^{q^{j-2}}, \dots,\ x_l^{q^{j-2}},\ y_l^{q^{j-2}}).$$

Since $\wr\wr \cdots \wr\wr$ is alternating we get

$$\begin{aligned}
0 &= \wr\wr F,\ F,\ F^{(3)}, \dots,\ F^{(n+2)} \wr\wr \\
&= \sum_{j=2}^{n+2} (-1)^j < F,\ F^{(j)} > ||F,\ F^{(3)}, \dots,\ \widehat{F^{(j)}}, \dots,\ F^{(n+2)}|| \\
&= \sum_{j=3}^{n+2} (-1)^j < F,\ F^{(j)} > \mathbf{E}_n \mathbf{d}_{n,\ j-2} \\
&= \sum_{j=3}^{n+2} (-1)^j \xi_{n,j-2} \mathbf{E}_n \mathbf{d}_{n,\ j-2} \\
&= \left(\sum_{j=3}^{n+2} (-1)^j \xi_{n,j-2} \mathbf{d}_{n,\ j-2} \right) \mathbf{E}_n.
\end{aligned}$$

Since $\mathbf{E}_n \neq 0 \in \mathbb{F}[V]^{\mathbb{Sp}(n,\ \mathbb{F})}$ we have

$$P_0 = \sum_{j=1}^{n} (-1)^j \xi_{n,j} \mathbf{d}_{n,j} = 0,$$

as claimed •

§4. Steenrod plays his game

We are going to calculate the Steenrod powers of our invariants, exploit Steenrod to find further relations, and to show that the $\mathbb{F}$-algebra generated by

$$\xi_{n,1}, \dots,\ \xi_{n,n},\ \mathbf{E}_n,\ \mathbf{d}_{n,\ 1}, \dots,\ \mathbf{d}_{n,\ n-1}$$

is closed under the action of the Steenrod algebra induced from $\mathbb{F}[V]$, i.e., we will show that

$$\mathrm{A} := \mathbb{F} < \xi_{n,1}, \dots,\ \xi_{n,n},\ \mathbf{E}_n,\ \mathbf{d}_{n,\ 1}, \dots,\ \mathbf{d}_{n,\ n-1} >$$

is an unstable algebra over the Steenrod algebra.

Since the Steenrod powers of the Dickson and Euler classes are known and as well polynomials in the Dickson and Euler classes, see [11] Appendix A.2 and the references there, we are left to deal with the new polynomials $\xi_{n,i}$. For simplicity of notation we make the following conventions:

CONVENTION: Let $\mathcal{P}^i \equiv 0$ whenever $i \notin \mathbb{N}_0$. Moreover, let $\xi_{n,i} = 0$ for $i \notin \mathbb{N}$.

LEMMA 4.1: *The Steenrod powers of the new classes $\xi_{n,i}$, $i \geq 1$, are given by the following formulae*

$$\mathcal{P}^j(\xi_{n,i}) = \begin{cases} \xi_{n,i}^q & \text{if } j = q^i + 1 \\ \xi_{n,i+1} & \text{if } j = q^i \\ \xi_{n,i-1}^q & \text{if } j = 1 \\ 0 & \text{otherwise} \end{cases} .$$

PROOF: By straightforward calculation:

$$\begin{aligned}
\mathcal{P}^j(\xi_{n,i}) &= \sum_{k=1}^{l} \left(\mathcal{P}^j(x_k y_k^{q^i}) - \mathcal{P}^j(x_k^{q^i} y_k)\right) \\
&= \sum_{k=1}^{l} \left(x_k \mathcal{P}^j(y_k^{q^i}) - \mathcal{P}^j(x_k^{q^i}) y_k + x_k^q \mathcal{P}^{j-1}(y_k^{q^i}) - \mathcal{P}^{j-1}(x_k^{q^i}) y_k^q\right) \\
&= \sum_{k=1}^{l} \left(x_k(\mathcal{P}^{j/q^i}(y_k)^{q^i} - \mathcal{P}^{j/q^i}(x_k)^{q^i} y_k + x_k^q \mathcal{P}^{(j-1)/q^i}(y_k)^{q^i} - \mathcal{P}^{(j-1)/q^i}(x_k)^{q^i} y_k^q\right) \\
&= \begin{cases} \sum_{k=1}^{l} x_k y_k^{q^{i+1}} - x_k^{q^{i+1}} y_k & \text{if } j = q^i \\ \sum_{k=1}^{l} x_k^q y_k^{q^{i+1}} - x_k^{q^{i+1}} y_k^q & \text{if } j = q^i + 1 \\ \sum_{k=1}^{l} x_k^q y_k^{q^i} - x_k^{q^i} y_k^q & \text{if } j = 1 \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}$$

which was to be shown •

We evaluate the Steenrod derivations on our ξ's in the next lemma.

LEMMA 4.2: *The Steenrod derivations act on the ξ's by*

$$\mathcal{P}^{\Delta_j}(\xi_{n,i}) = \begin{cases} (-1)^j \xi_{n,j-i}^{q^i} & \text{if } j > i \\ 0 & \text{if } j = i \\ (-1)^{j+1} \xi_{n,i-j}^{q^j} & \text{if } j < i \end{cases} .$$

PROOF: By induction on j. For $j = 1$ we have

$$\begin{aligned}
\mathcal{P}^{\Delta_1}(\xi_{n,i}) &= \mathcal{P}^1(\xi_{n,i}) \\
&= \begin{cases} 0 & \text{if } i = 1 \\ \xi_{n,i-1}^q & \text{if } i > 1 \end{cases} ,
\end{aligned}$$

where we made use of the preceding lemma. Next take an $j > 1$. Then we get by using again the preceding lemma and the induction hypothesis

$$\begin{aligned}
\mathcal{P}^{\Delta_j}(\xi_{n,i}) &= \mathcal{P}^{\Delta_{j-1}} \mathcal{P}^{q^{j-1}}(\xi_{n,i}) - \mathcal{P}^{q^{j-1}} \mathcal{P}^{\Delta_{j-1}}(\xi_{n,i}) \\
&= \begin{cases} 0 - (-1)^{j-1} \mathcal{P}^{q^{j-1}}\left(\xi_{n,j-1-i}^{q^i}\right) & \text{if } j - 1 > i \\ \mathcal{P}^{\Delta_{j-1}}(\xi_{n,i+1}) - 0 & \text{if } j - 1 = i \\ 0 - (-1)^j \mathcal{P}^{q^{j-1}}\left(\xi_{n,i-j+1}^{q^{j-1}}\right) & \text{if } j - 1 < i \end{cases}
\end{aligned}$$

$$= \begin{cases} (-1)^j \mathcal{P}^{q^{j-i-1}} \left(\xi_{n,j-1-i}\right)^{q^i} & \text{if } j-1 > i \\ (-1)^j \xi_{n,i+1-j+1}^{q^{j-1}} & \text{if } j-1 = i \\ (-1)^{j+1} \mathcal{P}^1 \left(\xi_{n,i-j+1}\right)^{q^{j-1}} & \text{if } j-1 < i \end{cases}$$

$$= \begin{cases} (-1)^j \xi_{n,j-i}^{q^i} & \text{if } j > i+1 \\ (-1)^j \xi_{n,1}^{q^{j-1}} & \text{if } j = i+1 \\ 0 & \text{if } j = i \\ (-1)^{j+1} \xi_{n,i-j}^{q^j} & \text{if } j < i \end{cases}$$

$$= \begin{cases} (-1)^j \xi_{n,j-i}^{q^i} & \text{if } j > i \\ 0 & \text{if } j = i \\ (-1)^{j+1} \xi_{n,i-j}^{q^j} & \text{if } j < i \end{cases},$$

as we wanted •

REMARK: Since

$$\mathcal{P}^{\Delta_i} := \begin{cases} \mathcal{P}^1 & \text{if } i = 1 \\ [\mathcal{P}^{q^i}, \mathcal{P}^{\Delta_{i-1}}] & \text{if } i > 1 \end{cases}$$

the the action of $\mathcal{P}^{\Delta_i}$ and $\mathcal{P}^{q^i}\mathcal{P}^{q^{i-1}} \cdots \mathcal{P}^q\mathcal{P}^1$ agree on the classes in V^*. This can be used to give another proof of Lemma 4.2.

Now we can prove

PROPOSITION 4.3: *The $\mathbb{F}$-algebra* A *generated by $\xi_{n,1}, \dots, \xi_{n,n}$, $\mathbf{E}_n$, $\mathbf{d}_{n,1}, \dots, \mathbf{d}_{n,n-1}$ is an unstable algebra over the Steenrod algebra.*

PROOF: The only thing we need to show is that our algebra A is closed under the action of the Steenrod algebra. The rest is inherited from $\mathbb{F}[V]$.

First note that part (1) of Proposition 3.2 tells us that

$$\mathbb{F} < \xi_{n,1}, \dots, \xi_{n,n-1}, \mathbf{d}_{n,1}, \dots, \mathbf{d}_{n,n-1} >$$

contains the Euler class $\mathbf{E}_n$, while Proposition 3.5 gives that it also contains $\xi_{n,n}$, i.e.,

$$\mathrm{A} = \mathbb{F} < \xi_{n,1}, \dots, \xi_{n,n-1}, \mathbf{d}_{n,1}, \dots, \mathbf{d}_{n,n-1} > .$$

From [11] Appendix A.2 we have for $j \geq 0$ and $i = 0, \dots, n-1$ that

$$\mathcal{P}^j(\mathbf{d}_{n,i}) \in \mathbb{F}[\mathbf{d}_{n,0}, \dots, \mathbf{d}_{n,n-1}] = \mathbb{F}[\mathbf{E}_n^{q-1}, \mathbf{d}_{n,1}, \dots, \mathbf{d}_{n,n-1}] \subseteq \mathrm{A}.$$

Next we consider $j \geq 0$ and $i = 1, \dots, n-1$ and get from Lemma 4.1

$$\mathcal{P}^j(\xi_{n,i}) \in \mathbb{F}[\xi_{n,1}, \dots, \xi_{n,n}] \subset \mathrm{A}$$

by Proposition 2.1 •

This allows us to construct a second family of relations from P_0 given in Proposition 3.5 in a very natural way.

COROLLARY 4.4: *In $\mathbb{F}[V]^{\mathbb{Sp}(n,\,\mathbb{F})}$ we have*

$$P_i := \sum_{j=i+1}^{n} \left((-1)^j \xi_{n,j-i}^{q^i} \mathbf{d}_{n,j}\right) - \sum_{j=0}^{i-1} \left((-1)^j \xi_{n,i-j}^{q^j} \mathbf{d}_{n,j}\right) = 0$$

for $i = 0, \dots, n-1$.

PROOF: By Proposition 3.5 we know that

$$P_0 = 0.$$

Therefore all Steenrod powers of this polynomial are zero, and, by Proposition 4.3, are again polynomials in the algebra generators of A. Observe that

$$P_{i+1} = \mathcal{P}^{q^i}(P_i),$$

which is proved by a straighforward calculation, to wit:

$$\begin{aligned}
\mathcal{P}^{q^i}(P_i) &= \sum_{j=i+1}^{n} \left((-1)^j \mathcal{P}^{q^i}\left(\xi_{n,j-i}^{q^i}\mathbf{d}_{n,j}\right)\right) - \sum_{j=0}^{i-1}\left((-1)^j \mathcal{P}^{q^i}\left(\xi_{n,i-j}^{q^j}\mathbf{d}_{n,j}\right)\right)\\
&= \sum_{j=i+1}^{n}(-1)^j\left(\sum_{\alpha+\beta=q^i}\mathcal{P}^{\alpha}(\xi_{n,j-i}^{q^i})\mathcal{P}^{\beta}(\mathbf{d}_{n,j})\right) - \sum_{j=0}^{i-1}(-1)^j\left(\sum_{\alpha+\beta=q^i}\mathcal{P}^{\alpha}(\xi_{n,i-j}^{q^j})\mathcal{P}^{\beta}(\mathbf{d}_{n,j})\right)\\
&= \sum_{j=i+1}^{n}(-1)^j\left(\sum_{\alpha+\beta=q^i}\mathcal{P}^{\frac{\alpha}{q^i}}(\xi_{n,j-i})^{q^i}\mathcal{P}^{\beta}(\mathbf{d}_{n,j})\right) - \sum_{j=0}^{i-1}(-1)^j\left(\sum_{\alpha+\beta=q^i}\mathcal{P}^{\frac{\alpha}{q^j}}(\xi_{n,i-j})^{q^j}\mathcal{P}^{\beta}(\mathbf{d}_{n,j})\right)\\
&= \sum_{j=i+1}^{n}(-1)^j\left(\mathcal{P}^{1}(\xi_{n,j-i})^{q^i}\mathbf{d}_{n,j} + \xi_{n,j-i}^{q^i}\mathcal{P}^{q^i}(\mathbf{d}_{n,j})\right)\\
&\quad - \sum_{j=0}^{i-1}(-1)^j\left(\xi_{n,i-j}^{q^j}\mathcal{P}^{q^i}(\mathbf{d}_{n,j}) + \mathcal{P}^{1}(\xi_{n,i-j})^{q^j}\mathcal{P}^{q^i-q^j}(\mathbf{d}_{n,j}) + \mathcal{P}^{q^{i-j}}(\xi_{n,i-j})^{q^j}\mathbf{d}_{n,j}\right)\\
&= \sum_{j=i+1}^{n}(-1)^j\left(\xi_{n,j-i-1}^{q^{i+1}}\mathbf{d}_{n,j}\right) + (-1)^{i+1}\xi_{n,1}^{q^i}\mathbf{d}_{n,i} - \sum_{j=0}^{i-1}(-1)^j\left(\xi_{n,i-j+1}^{q^j}\mathbf{d}_{n,j}\right)\\
&= \sum_{j=i+2}^{n}(-1)^j\left(\xi_{n,j-i-1}^{q^{i+1}}\mathbf{d}_{n,j}\right) - \sum_{j=0}^{i}(-1)^j\left(\xi_{n,i-j+1}^{q^j}\mathbf{d}_{n,j}\right)\\
&= P_{i+1},
\end{aligned}$$

where we made heavily use of the Cartan formulae, Lemma 4.1 and Appendix A.2 in [11] •

Note that Proposition 4.3 tells us *a priori* that all Steenrod powers of P_0 are again polynomials in the algebra generators of A. However, for their explicite description we had to calculate them anyway. In Section 6 it will turn out that we need only to consider $P_0, \dots, P_{l-1}$. Moreover, note that

$$P_i \equiv \xi_{n,n-i}^{q^i} \quad \textbf{MOD}\ (\xi_{n,1}, \dots, \xi_{n,n-i-1})$$

whenever $i \leq l-1$.

§5. British $\mathcal{T}$

Recall from Proposition 3.2 (1) that the Euler class $\mathbf{E}_n$ is a polynomial in $\xi_{n,1}, \dots, \xi_{n,n-1}$. So, there exists a polynomial $\mathcal{E}_n = \mathcal{E}_n(X_1, \dots, X_{n-1})$ such that

$$\mathbf{E}_n = \mathcal{E}_n(\xi_{n,1}, \dots, \xi_{n,n-1}).$$

The same proposition, part (2), shows that

$$\mathbf{E}_n\mathbf{d}_{n,\,i} \in \mathbb{F}[\xi_{n,1}, \dots, \xi_{n,n}],$$

i.e., there exist polynomials $\mathcal{D}_{n,i} = \mathcal{D}_{n,i}(X_1, \dots, X_n)$ such that

$$\mathbf{E}_n \mathbf{d}_{n,\ i} = \mathcal{D}_{n,i}(\xi_{n,1}, \dots, \xi_{n,n}),$$

for all $i = 0, \dots, n-1$. Moreover, we have seen that these polynomials $\mathcal{D}_{n,i}$ are linear in their last indeterminant, which has coefficient

$$\left(\mathcal{D}_{n-2,i-1}(X_1, \dots, X_{n-2}\right)^q,$$

compare Lemma 3.3.

PROPOSITION 5.1: *There exist polynomials*

$$\mathcal{T}_{i,j} = \mathcal{T}_{i,j}(X_1, \dots, X_n) \in \mathbb{F}[X_1, \dots, X_n]$$

such that for $i,\ j = 1, \dots, l$ *we have*

$$\mathcal{D}_{n,i} = \sum_{k=0}^{l-i} \mathcal{T}_{i,l-k-i+1}^{q^k} \mathcal{D}_{n,n-k}$$

and

$$\mathcal{T}_{i,j} = \mathcal{T}_{i,j}(X_1, \dots, X_{2(i+j)-3}) \in \mathbb{F}[X_1, \dots, X_{2(i+j)-3}],$$

i.e., $\mathcal{T}_{i,j}$ *depents only on the first* $2(i+j)-3$ *variables.*

PROOF: We construct the $\mathcal{T}_{i,j}$ by induction on j. Let $j = 1$. Then we define

$$\mathcal{T}_{i,1}(X_1, \dots, X_n) := \mathcal{E}_{2i}(X_1, \dots, X_n)^{q-1},$$

which is in $\mathbb{F}[X_1, \dots, X_{2i-1}]$, because the polynomial $\mathcal{E}_{2i}$ lives there. For $i = l$ this polynomial satisfies the desired relation

$$\mathcal{D}_{n,0} = \mathcal{E}_n^q = \mathcal{E}_n^{q-1}\mathcal{E}_n = \mathcal{T}_{l,1}\mathcal{E}_n.$$

For $i < l$ there is nothing more to prove.

Next take an $j > 1$ and assume $\mathcal{T}_{i,j}$ is defined for all $i = 1, \dots, l$ and all $j = 1, \dots, l-i$ such that the required relations hold. We then have by the induction hypothesis that

$$\mathcal{T}_{i,l-i+1}\mathcal{E}_n = \mathcal{D}_{n,l-i} - \sum_{k=1}^{l-i} \mathcal{T}_{i,l-k-i+1}^{q^k} \mathcal{D}_{n,n-k} \in \mathbb{F}[X_1, \dots, X_n].$$

We want to show that

$$\mathcal{T}_{i,l-i+1} \in \mathbb{F}[X_1, \dots, X_{n-1}].$$

From Proposition 3.2 (2) it follows that

$$\begin{aligned} \mathcal{T}_{i,l-i+1}\mathcal{E}_n &= \mathcal{D}_{n,l-i} - \sum_{k=1}^{l-i} \mathcal{T}_{i,l-k-i+1}^{q^k} \mathcal{D}_{n,n-k} \\ &= \mathcal{D}_{n,l-i}(\xi_{n,1}, \dots, \xi_{n,n}) - \sum_{k=1}^{l-i} \mathcal{T}_{i,l-k-i+1}^{q^k} \mathcal{D}_{n,n-k}(\xi_{n,1}, \dots, \xi_{n,n}) \end{aligned}$$

is an element in $\mathbb{F}[X_1, \dots, X_n]$. Since $k = 1, \dots, l-i$ we have by induction

$$\mathcal{T}_{i,l-k-i+1} \in \mathbb{F}[X_1, \dots, X_{2l-2k-1}].$$

Therefore, together with Lemma 3.3 our polynomial is linear in X_n with leading coefficient

$$\mathcal{D}_{n-2,l-i-1}(X_1,\ldots,X_{n-2})^q - \sum_{k=1}^{l-i}\mathcal{T}^{q^k}_{i,l-k-i+1}\mathcal{D}_{n-2,n-k-1}(X_1,\ldots,X_{n-2})^q =$$

$$\left(\mathcal{D}_{n-2,l-i-1}(X_1,\ldots,X_{n-2}) - \sum_{k=0}^{l-i-1}\mathcal{T}^{q^k}_{i,l-k-i}\mathcal{D}_{n-2,n-2-k}(X_1,\ldots,X_{n-2})\right)^q$$

$$= 0,$$

where the last equation follows from the induction hypothesis. Therefore

$$\mathcal{T}_{i,l-i+1}\mathcal{E}_n \in \mathbb{F}[X_1,\ldots,X_{n-1}]$$

which in turn implies that

$$\mathcal{T}_{i,l-i+1} \in \mathbb{F}[X_1,\ldots,X_{n-1}],$$

where the desired relation holds by construction •

§6. Some Algebra

In this section we do some algebra and prove that we have found all generators and relations of our ring of invariants. By what we have done so far we know that

$$A = \mathbb{F} < \xi_{n,1},\ldots,\xi_{n,n}, \mathbf{d}_{n,0},\ldots,\mathbf{d}_{n,n-1} >= \mathbb{F} < \xi_{n,1},\ldots,\xi_{n,n}, \mathbf{d}_{n,1},\ldots,\mathbf{d}_{n,n-1} >,$$

compare Proposition 3.2 (1).[10]

Next we show that we can omit the Dickson classes of high degree.

LEMMA 6.1: *With the preceding notation we have*

$$A = \mathbb{F} < \xi_{n,1},\ldots,\xi_{n,n}, \mathbf{d}_{n,l},\ldots,\mathbf{d}_{n,n-1} > .$$

PROOF: Certainly A contains this algebra. So we have to show that

$$\mathbf{d}_{n,1},\ldots,\mathbf{d}_{n,l-1} \in \mathbb{F} < \xi_{n,1},\ldots,\xi_{n,n}, \mathbf{d}_{n,l},\ldots,\mathbf{d}_{n,n-1} > .$$

Proposition 5.1 hands us equations

$$\mathbf{d}_{n,l-i} = \sum_{k=0}^{l-i}\mathcal{T}^{q^k}_{i,l-i-k+1}(\xi_{n,1},\ldots,\xi_{n,2(l-k)-1})\mathbf{d}_{n,n-k},$$

for $i = 1,\ldots,l-1$. Since $2l-2k-1 \leq 2l-1$ for $k \geq 0$ we get,

$$\mathbf{d}_{n,l-i} = \sum_{k=0}^{l-i}\mathcal{T}^{q^k}_{i,l-i-k+1}(\xi_{n,1},\ldots,\xi_{n,2(l-k)-1})\mathbf{d}_{n,n-k}, \in \mathbb{F} < \xi_{n,1},\ldots,\xi_{n,n}, \mathbf{d}_{n,l},\ldots,\mathbf{d}_{n.n-1} >,$$

where we use that $i = 1,\ldots,l-1$ •

Consider the remembering map ϱ

$$\varrho : B := \mathbb{F}[\xi_{n,1},\ldots,\xi_{n,n}, \mathbf{d}_{n,l},\ldots,\mathbf{d}_{n,n-1}] \longrightarrow \mathbb{F}[V]^{\mathbb{Sp}(n,\,\mathbb{F})}.$$

---

[10] We forget for a moment that P_0, given in Proposition 3.5, cancels $\xi_{n,n}$.

Recall the remarks after Corollary 4.4

$$P_i \equiv \xi_{n,n-i}^{q^i} \ \mathbf{MOD}(\xi_{n,1}, \ldots, \xi_{n,n-i-1}),$$

whenever $i \leq l-1$. Since the sequence

$$\xi_{n,1}, \ldots, \xi_{n,n}, \ \mathbf{d}_{n,l}, \ldots, \mathbf{d}_{n,n-1} \in \mathrm{B}$$

forms a regular sequence, so does the sequence

$$\xi_{n,1}, \ldots, \xi_{n,n-1}, \ P_0, \ \mathbf{d}_{n,l}, \ldots, \mathbf{d}_{n,n-1} \in \mathrm{B}.$$

Therefore also $\xi_{n,1}, \ldots, \xi_{n,n-1}^{q}, \ P_0, \ \mathbf{d}_{n,l}, \ldots, \mathbf{d}_{n,n-1} \in \mathrm{B}$ forms a regular sequence and hence so does $\xi_{n,1}, \ldots, \xi_{n,n-2}, \ P_1, \ P_0, \ \mathbf{d}_{n,l}, \ldots, \mathbf{d}_{n,n-1} \in \mathrm{B}$. Successively we get that

$$\xi_{n,1}, \ldots, \xi_{n,l}, \ P_{l-1}, \ldots, P_0, \ \mathbf{d}_{n,l}, \ldots, \mathbf{d}_{n,n-1} \in \mathrm{B}$$

is a regular sequence, and in particular

$$P_0, \ldots, P_{l-1} \in \mathrm{B}$$

is a regular sequence. Hence we have shown

LEMMA 6.2: *The* $\mathbb{F}$*-algebra*

$$\mathrm{B}/\left(P_0, \ldots, P_{l-1}\right) = \mathbb{F}[\xi_{n,1}, \ldots, \xi_{n,n}, \ \mathbf{d}_{n,l}, \ldots, \mathbf{d}_{n,n-1}]/\left(P_0, \ldots, P_{l-1}\right)$$

is a complete intersection of Krull dimension n*. In particular it is a Cohen-Macaulay algebra.*

PROOF: The Cohen-Macaulayness follows from the same calculation:

$$\xi_{n,1}, \ldots, \xi_{n,l}, \ P_{l-1}, \ldots, P_0, \ \mathbf{d}_{n,l}, \ldots, \mathbf{d}_{n,n-1} \in \mathrm{B}$$

is a regular sequence, and hence in the quotient algebra

$$\xi_{n,1}, \ldots, \xi_{n,l}, \ \mathbf{d}_{n,l}, \ldots, \mathbf{d}_{n,n-1} \in \mathrm{B}/\left(P_0, \ldots, P_{l-1}\right),$$

is a regular sequence of length n •

What we are about to do is to show that this algebra $\mathrm{B}/\left(P_0, \ldots, P_{l-1}\right)$ is precisely the ring of invariants we are looking for, and, moreover the same as A.

The proof of the following lemma uses Nagata's theorem, see [9], i.e., one of the few existing standard methods to prove that a given ring is the desired ring of invariants, compare [10].

LEMMA 6.3: *The algebra* $\mathrm{B}/\left(P_0, \ldots, P_{l-1}\right)$ *is a unique factorization domain.*

PROOF: We rewrite our system of relations $P_0, \ldots, P_{l-1}$ as a system of linear equations for the Dickson classes, i.e., the system

$$\begin{array}{rcl} P_0 & = & 0 \\ \ddots & & \ddots \\ P_i & = & 0 \\ \ddots & & \ddots \\ P_{l-1} & = & 0 \end{array}$$

is by Corollary 4.4 equivalent to

$$\begin{array}{rcl}
\sum_{j=1}^{n}\left((-1)^j\xi_{n,j}\mathbf{d}_{n,j}\right) & = & 0 \\
\ddots & & \ddots \\
\sum_{j=i+1}^{n}\left((-1)^j\xi_{n,j-i}^{q^i}\mathbf{d}_{n,j}\right)-\sum_{j=0}^{i-1}\left((-1)^j\xi_{n,i-j}^{q^j}\mathbf{d}_{n,j}\right) & = & 0 \\
\ddots & & \ddots \\
\sum_{j=l}^{n}\left((-1)^j\xi_{n,j-l+1}^{q^{l-1}}\mathbf{d}_{n,j}\right)-\sum_{j=0}^{l-2}\left((-1)^j\xi_{n,l-1-j}^{q^j}\mathbf{d}_{n,j}\right) & = & 0,
\end{array}$$

what in turn can be written as

$$\begin{array}{rcl}
\sum_{j=l}^{n-1}\left((-1)^j\xi_{n,j}\mathbf{d}_{n,j}\right)+\sum_{j=1}^{l-1}\left((-1)^j\xi_{n,j}\mathbf{d}_{n,j}\right)+(-1)^n\xi_{n,n} & = & 0 \\
\ddots & & \ddots \\
\sum_{j=l}^{n-1}\left((-1)^j\xi_{n,j-i}^{q^i}\mathbf{d}_{n,j}\right)+\sum_{j=i+1}^{l-1}\left((-1)^j\xi_{n,j-i}^{q^i}\mathbf{d}_{n,j}\right)-\sum_{j=0}^{i-1}\left((-1)^j\xi_{n,i-j}^{q^j}\mathbf{d}_{n,j}\right)+(-1)^n\xi_{n,n-i}^{q^i} & = & 0 \\
\ddots & & \ddots \\
\sum_{j=l}^{n-1}\left((-1)^j\xi_{n,j-l+1}^{q^{l-1}}\mathbf{d}_{n,j}\right)-\sum_{j=0}^{l-2}\left((-1)^j\xi_{n,l-1-j}^{q^j}\mathbf{d}_{n,j}\right)+(-1)^n\xi_{n,n-l+1}^{q^{l-1}} & = & 0.
\end{array}$$

We set

$$\mathbf{M}=\begin{bmatrix}
(-1)^l\xi_{n,l} & \cdots & (-1)^{n-1}\xi_{n,n-1} \\
\cdots & \cdots & \cdots \\
(-1)^l\xi_{n,l-i}^{q^i} & \cdots & (-1)^{n-1}\xi_{n,n-1-i}^{q^i} \\
\cdots & \cdots & \cdots \\
(-1)^l\xi_{n,1}^{q^{l-1}} & \cdots & (-1)^{n-1}\xi_{n,l}^{q^{l-1}}
\end{bmatrix}$$

and

$$\mathbf{N}=\begin{bmatrix}
0 & \xi_{n,1} & \cdots & \cdots & (-1)^{i-1}\xi_{n,i-1} & \cdots & \cdots & (-1)^l\xi_{n,l-1} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\xi_{n,i} & -\xi_{n,i-1}^{q} & \cdots & (-1)^{i-1}\xi_{n,1}^{q^{i-1}} & 0 & (-1)^i\xi_{n,1}^{q^i} & \cdots & (-1)^l\xi_{n,l-1-i}^{q^i} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\xi_{n,l-1}^{q} & -\xi_{n\,l-2}^{q} & \cdots & (-1)^i\xi_{n,l-i}^{q^{i-1}} & \cdots & \cdots & (-1)^{l-2}\xi_{n,1}^{q^{l-2}} & 0
\end{bmatrix}$$

and get a system of linear equations as follows

$$\mathbf{M}\begin{bmatrix}\mathbf{d}_{n,l}\\ \ddots \\ \mathbf{d}_{n,n-1}\end{bmatrix}-\mathbf{N}\begin{bmatrix}\mathbf{d}_{n,0}\\ \ddots \\ \mathbf{d}_{n,l-1}\end{bmatrix}+(-1)^n\begin{bmatrix}\xi_{n,n}\\ \ddots \\ \xi_{n,n-l+1}^{q^{l-1}}\end{bmatrix}=\begin{bmatrix}0\\ \ddots \\ 0\end{bmatrix}.$$

Recall from Proposition 5.1 that the Dickson classes of high degree are given by

$$\begin{aligned}
\mathbf{d}_{n,j} &= \sum_{k=0}^{j} T_{l-j,j-k+1}^{q^k}(\xi_{n,1},\ldots,\xi_{n,2(l-k)-1})\mathbf{d}_{n,n-k} \\
&= T_{l-j,j+1}(\xi_{n,1},\ldots,\xi_{n,2l-1})+\sum_{k=1}^{j} T_{l-j,j-k+1}^{q^k}(\xi_{n,1},\ldots,\xi_{n,2(l-k)-1})\mathbf{d}_{n,n-k}.
\end{aligned}$$

We put that in our system of linear equations, set $\mathcal{T}_{i,j}(\xi) := \mathcal{T}_{i,j}(\xi_{n,1}, \dots, \xi_{n,2(i+j)-1})$ for short and get

$$\begin{aligned}
\begin{bmatrix} 0 \\ \ddots \\ 0 \end{bmatrix} &= \mathbf{M} \begin{bmatrix} \mathbf{d}_{n,l} \\ \ddots \\ \mathbf{d}_{n,n-1} \end{bmatrix} - \mathbf{N} \begin{bmatrix} \mathbf{d}_{n,0} \\ \ddots \\ \mathbf{d}_{n,l-1} \end{bmatrix} + (-1)^n \begin{bmatrix} \xi_{n,n} \\ \ddots \\ \xi_{n,n-l+1}^{q^{l-1}} \end{bmatrix} \\
&= \mathbf{M} \begin{bmatrix} \mathbf{d}_{n,l} \\ \ddots \\ \mathbf{d}_{n,n-1} \end{bmatrix} - \mathbf{N} \begin{bmatrix} \mathbf{d}_{n,0} \\ \mathcal{T}_{l-1,2}(\xi) \\ \ddots \\ \mathcal{T}_{1,l}(\xi) \end{bmatrix} - \mathbf{N} \begin{bmatrix} 0 \\ \mathcal{T}^{q}_{l-1,1}(\xi) \\ \ddots \\ \mathcal{T}^{q}_{1,l-1}(\xi) \end{bmatrix} \mathbf{d}_{n,n-1} - \cdots \\
&\quad \cdots - \mathbf{N} \begin{bmatrix} 0 \\ \ddots \\ 0 \\ \mathcal{T}^{q^{l-1}}_{1,1}(\xi) \end{bmatrix} \mathbf{d}_{n,l+1} + (-1)^n \begin{bmatrix} \xi_{n,n} \\ \ddots \\ \xi_{n,n-l+1}^{q^{l-1}} \end{bmatrix}.
\end{aligned}$$

Denote by

$$\mathbf{L} := \begin{bmatrix} \begin{matrix} 0 \\ \ddots \\ \ddots \\ 0 \end{matrix} & \left(\mathbf{N} \begin{bmatrix} 0 \\ \ddots \\ 0 \\ \mathcal{T}^{q^{l-1}}_{1,1}(\xi) \end{bmatrix} \right) & \cdots & \left(\mathbf{N} \begin{bmatrix} 0 \\ \mathcal{T}^{q}_{l-1,1}(\xi) \\ \ddots \\ \mathcal{T}^{q}_{1,l-1}(\xi) \end{bmatrix} \right) \end{bmatrix}$$

the $l \times l$ matrix with columns

$$\begin{pmatrix} 0 \\ \ddots \\ \ddots \\ 0 \end{pmatrix} \quad \left(\mathbf{N} \begin{bmatrix} 0 \\ \ddots \\ 0 \\ \mathcal{T}^{q^{l-1}}_{1,1}(\xi) \end{bmatrix} \right) \quad \cdots \quad \left(\mathbf{N} \begin{bmatrix} 0 \\ \mathcal{T}^{q}_{l-1,1}(\xi) \\ \ddots \\ \mathcal{T}^{q}_{1,l-1}(\xi) \end{bmatrix} \right).$$

Then

$$\begin{aligned}
\begin{bmatrix} 0 \\ \ddots \\ 0 \end{bmatrix} &= \mathbf{M} \begin{bmatrix} \mathbf{d}_{n,l} \\ \ddots \\ \mathbf{d}_{n,n-1} \end{bmatrix} - \mathbf{N} \begin{bmatrix} \mathbf{d}_{n,0} \\ \ddots \\ \mathbf{d}_{n,l-1} \end{bmatrix} + (-1)^n \begin{bmatrix} \xi_{n,n} \\ \ddots \\ \xi_{n,n-l+1}^{q^{l-1}} \end{bmatrix} \\
&= \mathbf{M} \begin{bmatrix} \mathbf{d}_{n,l} \\ \ddots \\ \mathbf{d}_{n,n-1} \end{bmatrix} - \mathbf{L} \begin{bmatrix} \mathbf{d}_{n,l} \\ \ddots \\ \mathbf{d}_{n,n-1} \end{bmatrix} \\
&\quad - \mathbf{N} \begin{bmatrix} \mathbf{d}_{n,0} \\ \mathcal{T}_{l-1,2}(\xi) \\ \ddots \\ \mathcal{T}_{1,l}(\xi) \end{bmatrix} + (-1)^n \begin{bmatrix} \xi_{n,n} \\ \ddots \\ \xi_{n,n-l+1}^{q^{l-1}} \end{bmatrix} \\
&= \mathbf{T} \begin{bmatrix} \mathbf{d}_{n,l} \\ \ddots \\ \mathbf{d}_{n,n-1} \end{bmatrix} - \mathbf{N} \begin{bmatrix} \mathcal{E}_n(\xi_{n,1}, \dots, \xi_{n,n-1})^{q-1} \\ \mathcal{T}_{l-1,2}(\xi) \\ \ddots \\ \mathcal{T}_{1,l}(\xi) \end{bmatrix} + (-1)^n \begin{bmatrix} \xi_{n,n} \\ \ddots \\ \xi_{n,n-l+1}^{q^{l-1}} \end{bmatrix},
\end{aligned}$$

where we set

$$\mathbf{T} := \mathbf{M} - \mathbf{L}$$

and use that the top Dickson class $\mathbf{d}_{n,0}$ is nothing but the $(q-1)$-st power of the Euler class $\mathbf{E}_n$ what in turn is a polynomial in the first $n-1$ ξ's

$$\mathbf{d}_{n,0} = (\mathbf{E}_n)^{q-1} = \mathcal{E}_n(\xi_{n,1}, \ldots, \xi_{n,n-1})^{q-1}$$

by Proposition 3.2. The matrix $\mathbf{T}$ is modulo $\xi_{n,1}, \ldots, \xi_{n,l-1}$ upper triangular with determinant

$$\det(\mathbf{T}) \equiv (-1)^{\frac{(3l-1)l}{2}} \xi_{n,l} \xi_{n,l}^{q} \cdots \xi_{n,l}^{q^{l-1}} =: \Delta \quad \textbf{MOD}\ \xi_{n,1}, \ldots, \xi_{n,l-1}.$$

By Lemma 6.2 we know that

$$\xi_{n,l} \in \mathrm{B}/\left(P_0, \ldots, P_{l-1}\right)$$

is not a zero divisor, hence so is Δ. Therefore we get by localizing at Δ an *inclusion*

$$\mathrm{B}/\left(P_0, \ldots, P_{l-1}\right) \hookrightarrow \mathrm{B}[\Delta^{-1}]/\left(P_0, \ldots, P_{l-1}\right).$$

In the bigger algebra the system of equations given by the relations $P_0, \ldots, P_{l-1}$ can be *solved* for $\mathbf{d}_{n,l}, \ldots, \mathbf{d}_{n,n-1}$ (well, we just inverted the determinant of the matrix of our system of linear equations) and hence

$$\mathrm{B}[\Delta^{-1}]/\left(P_0, \ldots, P_{l-1}\right) = \mathbb{F}[\xi_{n,1}, \ldots, \xi_{n,n}, \Delta^{-1}]$$

is a polynomial algebra, and in particular an integral domain. Hence

$$\mathrm{B}/\left(P_0, \ldots, P_{l-1}\right)$$

is also an integral domain.

Next, we want to show that Δ is a prime element in $B/(P_0, \ldots, P_{l-1})$. Observe that the entries of the matrix $\mathbf{T}$ are polynomials in $\xi_{n,1}, \ldots, \xi_{n,n-1}$. However, $\xi_{n,n-1}$ occurs only in the top right corner, i.e., we can rewrite $\mathbf{T}$ as

$$\mathbf{T} := \begin{bmatrix} (-1)^{l}\xi_{n,l} & \cdots & (-1)^{n-2}\xi_{n,n-2} & (-1)^{n-1}\xi_{n,n-1} \\ & & & (-1)^{n-1}\xi_{n,n-2}^{q} + N_1 \\ & \mathbf{T}^{\mathbf{cof}} & & \ddots \\ & & & (-1)^{n-1}\xi_{n,l}^{q^{l-1}} + N_{l-1} \end{bmatrix},$$

where we set

$$\mathbf{N}\begin{bmatrix} 0 \\ \mathcal{T}_{l-1,1}^{q}(\xi) \\ \ddots \\ \mathcal{T}_{1,l-1}^{q} \end{bmatrix}(\xi) = \begin{bmatrix} 0 \\ N_1 \\ \ddots \\ N_{l-1} \end{bmatrix}.$$

The cofactor matrix $\mathbf{T}^{\mathbf{cof}}$ has determinant

$$\Delta^{\mathbf{cof}} \in \mathbb{F}[\xi_{n,1}, \ldots, \xi_{n,n-2}].$$

Therefore

$$\Delta^{\mathbf{cof}} \in B/(P_0, \ldots, P_{l-1}, \Delta)$$

is not a zero divisor (with a little help from Lemma 6.2). So, if we invert this determinant we get an *inclusion*

$$\mathrm{B}/(P_0, \ldots, P_{l-1}, \Delta) \hookrightarrow \mathrm{B}[(\Delta^{\mathbf{cof}})^{-1}]/(P_0, \ldots, P_{l-1}, \Delta).$$

In the localization the relations $P_1, \ldots, P_{l-1}$ can be solved for $\mathbf{d}_{n,l}, \ldots, \mathbf{d}_{n,n-2}$ and P_0 gives an equation for $\xi_{n,n}$. The determinant Δ is linear in $\xi_{n,n-1}$ and since we inverted the leading coefficient, namely $\Delta^{\mathbf{cof}}$, Δ can be solved for $\xi_{n,n-1}$. Hence

$$\mathrm{B}[(\Delta^{\mathbf{cof}})^{-1}]/(P_0, \ldots, P_{l-1}, \ \Delta). = \mathbb{F}[\xi_{n,1}, \ldots, \xi_{n,n-2}, \ \mathbf{d}_{n,n-1}, \ (\Delta^{\mathbf{cof}})^{-1}]$$

is a polynomial ring and in particular an integral domain. Therefore the little algebra

$$\mathrm{B}/(P_0, \ldots, P_{l-1}, \ \Delta)$$

is an integral domain, what in turn implies that

$$\Delta \in \mathrm{B}/(P_0, \ldots, P_{l-1})$$

is a prime element. Therefore

$$\mathrm{B}/(P_0, \ldots, P_{l-1})$$

is a unique factorization domain, because its localization at Δ

$$\mathrm{B}[\Delta^{-1}]/\left(P_0, \ldots, P_{l-1}\right) = \mathbb{F}[\xi_{n,1}, \ldots, \xi_{n,n}, \ \Delta^{-1}]$$

is, where we use Nagata's wonderful theorem, [9] or [2] Lemma 2.2.2. •

Finally, we are going to be rewarded with the explicit description of the ring of invariants of the symplectic group.

THEOREM 6.4: *With the preceding notation, the ring of invariants $\mathbb{F}[V]^{\mathbb{Sp}(n, \mathbb{F})}$ is given by*

$$\mathrm{B}/\left(P_0, \ldots, P_{l-1}\right) = \mathbb{F}[\xi_{n,\,1}, \ldots, \xi_{n,\,n-1}, \ \mathbf{d}_{n,\,l}, \ldots, \mathbf{d}_{n,\,n-1}]/\left(P_1, \ldots, P_{l-1}\right).$$

PROOF: We have the remembering map

$$\varrho : \mathrm{B} = \mathbb{F}[\xi_{n,\,1}, \ldots, \xi_{n,\,n-1}, \ \mathbf{d}_{n,\,l}, \ldots, \mathbf{d}_{n,\,n}] \longrightarrow \mathbb{F}[V]^{\mathbb{Sp}(n,\,\mathbb{F})}.$$

By Corollary 4.4 the kernel contains the polynomials $P_0, \ldots, P_{l-1}$. Hence ϱ factorizes through

$$\varphi : \mathrm{B}/\left(P_0, \ldots, P_{l-1}\right) \longrightarrow \mathbb{F}[V]^{\mathbb{Sp}(n,\,\mathbb{F})}.$$

In Lemma 6.3 we have seen that the quotient

$$\mathrm{B}/\left(P_0, \ldots, P_{l-1}\right)$$

is an integral domain, so the map φ is injective. By construction we have that

$$\mathscr{D}^*(n) \hookrightarrow \mathrm{Im}(\varphi) \hookrightarrow \mathbb{F}[V]^{\mathbb{Sp}(n,\,\mathbb{F})}.$$

Since the over all ring extension is finite, so is

$$\mathrm{Im}(\varphi) \hookrightarrow \mathbb{F}[V]^{\mathbb{Sp}(n,\,\mathbb{F})},$$

and in particular integral. So, let's have a look at what we have now:

$$\mathscr{D}^*(n) \hookrightarrow \mathrm{B}/\left(P_0, \ldots, P_{l-1}\right) \cong \mathrm{Im}(\varphi) \underset{\text{integral, finite}}{\hookrightarrow} \mathbb{F}[V]^{\mathbb{Sp}(n,\,\mathbb{F})} \underset{\text{integral, finite}}{\hookrightarrow} \mathbb{F}[V]$$

and at the level of fields of fractions:

$$FF(\mathscr{D}^*(n)) \hookrightarrow FF(\mathrm{B}/\left(P_0, \ldots, P_{l-1}\right)) \cong FF(\mathrm{Im}(\varphi)) \underset{\text{finite}}{\hookrightarrow} \mathbb{F}(V)^{\mathbb{Sp}(n,\,\mathbb{F})} \underset{\text{Galois}}{\hookrightarrow} \mathbb{F}[V].$$

So, we have Galois groups as follows

$$\mathrm{Gal}\left(\mathbb{F}[V]/FF(\mathcal{D}^*(n))\right) = \mathrm{GL}(n,\ \mathbb{F}) \supseteq \mathrm{Gal}\left(\mathbb{F}[V]/FF(\mathrm{Im}(\varphi))\right) \supseteq \mathrm{Gal}\left(\mathbb{F}[V]/\mathbb{F}(V)^{\mathbb{Sp}(n,\ \mathbb{F})}\right) = \mathbb{Sp}(n,\ \mathbb{F}).$$

Since the image of φ contains $\xi_{n,1}$, so does its field of fractions, i.e.,

$$\mathrm{Gal}\left(\mathbb{F}[V]/FF(\mathrm{Im}(\varphi))\right) \subseteq \mathrm{GL}(n,\ \mathbb{F})_{\xi_{n,1}} = \mathbb{Sp}(n,\ \mathbb{F}),$$

where we made use of Lemma 2.2. That means that our field of fractions is correct:

$$\begin{tikzcd}
\mathrm{Im}(\varphi) \arrow[r, hook, "\text{integral}"'] \arrow[d, hook] & F[V]^{\mathbb{Sp}(n,\ \mathbb{F})} \arrow[d, hook] \\
FF(\mathrm{Im}(\varphi)) \arrow[r, phantom, "="] & \mathbb{F}(V)^{\mathbb{Sp}(n,\ \mathbb{F})}.
\end{tikzcd}$$

Moreover, $\mathbb{F}[V]^{\mathbb{Sp}(n,\ \mathbb{F})}$ is integrally closed, because its a ring of invariants (well, that goes back to Emmy) and $\mathrm{Im}(\varphi)$ is integrally closed, because its a unique factorization domain, as we had seen in Lemma 6.3, [14] Example 1 of Section V.3. Since the ring extension is integral we have that both rings must be equal •

Finally let's just summarize what we know about the symplectic invariants. First of all they are explicitely given by

$$\mathbb{F}[V]^{\mathbb{Sp}(n,\ \mathbb{F})} = \mathbb{F}[\xi_{n,1}, \ldots, \xi_{n,n-1}, \mathbf{d}_{n,l}, \ldots, \mathbf{d}_{n,n-1}]/\left(P_1, \ldots, P_{l-1}\right),$$

where the generators are given by Proposition 2.1

$$\xi_{n,\ i} := \sum_{j=1}^{l}(x_j y_j^{q^i} - y_j x_j^{q^i}) \in \mathbb{F}[x_1,\ y_1, \ldots, x_l,\ y_l]^{\mathbb{Sp}(n,\ \mathbb{F})},$$

while the $\mathbf{d}_{n,i}$'s are the Dickson classes of degree $q^n - q^i$. The relations are explicity given by Proposition 3.5 and Corollary 4.4

$$P_i := \sum_{j=i+1}^{n}\left((-1)^j \xi_{n,j-i}^{q^i} \mathbf{d}_{n,j}\right) - \sum_{j=0}^{i-1}\left((-1)^j \xi_{n,i-j}^{q^j} \mathbf{d}_{n,j}\right)$$

for $i = 1, \ldots, l-1$. Moreover the ring has Krull dimension n and is an integral domain, which is no surprise, since any ring of invariants has these properties. What is not always the case but holds for the ring of invariants of the symplectic group is: it is a unique factorization domain by Lemma 6.3, and a complete intersection by Lemma 6.2 (and a fortiori Cohen-Macaulay), i.e., its still a relatively nice ring.

§7. Examples and Rational Invariants

We want to calculate in this final section also the rational invariants of the symplectic group, what is pretty easy now after all this hard work. Moreover, we have a look at some examples.

THEOREM 7.1: *The rational invariants of the symplectic group are given by*

$$\mathbb{F}(V)^{\mathbb{Sp}(n,\ \mathbb{F})} = \mathbb{F}(\xi_{n,1}, \ldots, \xi_{n,n}).$$

PROOF: By Theorem 6.4

$$\mathbb{F}(V)^{\mathbb{Sp}(n,\ \mathbb{F})} = FF\left(\mathbb{F}[\xi_{n,\ 1}, \dots, \xi_{n,\ n},\ \mathbf{d}_{n,\ l}, \dots, \mathbf{d}_{n,\ n-1}]/\left(P_0, \dots, P_{l-1}\right)\right),$$

(Note carefully that I put back in the last $\xi_{n,\ n}$ which is compensated by adding the respective relation P_0.) However, in the field of fractions, the relations $P_1, \dots, P_{l-1}$ can be solved for $\mathbf{d}_{n,\ l}, \dots, \mathbf{d}_{n,\ n-1}$, while P_0 becomes the trivial relation, i.e., with the help of $P_1, \dots, P_{l-1}$ the Dickson polynomials can be expressed as rational functions in $\xi_{n,1}, \dots, \xi_{n,n}$. So we have[11]

$$\mathbb{F}(V)^{\mathbb{Sp}(n,\ \mathbb{F})} = \mathbb{F}(\xi_{n,\ 1}, \dots, \xi_{n,\ n})$$

and this is indeed neat •

Next we look at examples. If $n = 2$ then the symplectic group is nothing else than the special linear group

$$\mathbb{Sp}(2,\ \mathbb{F}) = \mathrm{SL}(2,\ \mathbb{F}),$$

(do Exercise 8.13 of [12] if you have doubts). So we knew the invariants from Dickson's work, namely

$$\mathbb{F}[V]^{\mathbb{Sp}(2,\ \mathbb{F})} = \mathbb{F}[\mathbf{E}_2,\ \mathbf{d}_{2,1}],$$

which you, of course, can look up in the bible: Theorem 8.1.8 in [13].

So, a bit more interesting is the next case: $n = 4$, and let's take the field with 3 elements. Then our symplectic group has order

$$51840 = 2^7 3^4 5,$$

by Dickson's calculations. The group is generated by the matrices

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

(recall Dickson's Theorem 1.2 from the first Section) and the proof of Satz 9.25 in [7] tells you that the bad guy, i.e., the 3-Sylow sub group is non Abelian and consists of matrices of the form

$$\begin{bmatrix} 1 & d & a & b \\ 0 & 1 & 0 & 0 \\ 0 & b-ac & 1 & c \\ 0 & -a & 0 & 1 \end{bmatrix}.$$

So, our ring of invariants is generated by

$$\xi_{4,1}, \dots, \xi_{4,4},\ \mathbf{E}_4,\ \mathbf{d}_{4,1},\ \mathbf{d}_{4,2},\ \mathbf{d}_{4,3}.$$

The relations given by Proposition 3.2 are[12]

$$\begin{aligned} \mathbf{E}_4 &= \xi_{4,1}^3 \xi_{4,3} - \xi_{4,2}^4 + \xi_{4,1}^{10} \\ \mathbf{E}_4 \mathbf{d}_{4,1} &= \xi_{4,2} \xi_{4,1}^{27} + \xi_{4,1}^9 \xi_{4,4} - \xi_{4,2}^9 \xi_{4,3} \\ \mathbf{E}_4 \mathbf{d}_{4,2} &= \xi_{4,1}^{28} + \xi_{4,2}^3 \xi_{4,4} - \xi_{4,3}^4 \\ \mathbf{E}_4 \mathbf{d}_{4,3} &= \xi_{4,1} \xi_{4,2}^9 - \xi_{4,1}^3 \xi_{4,4} - \xi_{4,3}^3 \xi_{4,2}. \end{aligned}$$

---

[11] Note, that this is purely trancendental over $\mathbb{F}$.

[12] The proof of this proposition gives you an explicite algorithm to find these expressions: note first that for $n = 4$ we have that $\tau = (12)(34)$, the centralizer $C(\tau) = \{e,\ (12),\ (34),\ (12)(34),\ (13)(24),\ (1324),\ (1423),\ (14)(23)\}$ and a complete set of coset representatives in Σ_4 is given by e, (13) and (14). So, if you use the formula given in Proposition 3.2 you get the above expression.

Note that $\mathbf{E}_4$ is a polynomial in $\xi_{4,1},\ \xi_{4,2},\ \xi_{4,3}$ as Proposition 3.2 (1) predicts. The relation P_0 given in Proposition 3.5 reads as follows

$$P_0 = \xi_{4,1}\mathbf{d}_{4,1} - \xi_{4,2}\mathbf{d}_{4,2} + \xi_{4,3}\mathbf{d}_{4,3} - \xi_{4,4} = 0.$$

Corollary 4.4 hands us the remaining P_1, which is

$$P_1 = \xi_{4,1}^{q}\mathbf{d}_{4,2} - \xi_{4,2}^{q}\mathbf{d}_{4,3} + \xi_{4,3}^{q} - \xi_{4,1}\mathbf{d}_{4,0} = 0.$$

The british $\mathcal{T}$'s of Proposition 5.1, evaluated at $\xi_{4,1}, \ldots, \xi_{4,4}$ are given by

$$\begin{aligned} \mathcal{T}_{1,1} &= \xi_{4,1}^{2} \\ \mathcal{T}_{1,2} &= \mathbf{d}_{4,1} - \xi_{4,1}^{6}\mathbf{d}_{4,3} = \xi_{4,1}^{27}\xi_{4,2} - \xi_{4,2}^{9}\xi_{4,3} - \xi_{4,1}^{7}\xi_{4,2}^{9} + \xi_{4,1}^{6}\xi_{4,2}\xi_{4,3}^{3} \\ \mathcal{T}_{2,1} &= \mathbf{d}_{4,0} = \mathbf{E}_4^2 = \left(\xi_{4,1}^{3}\xi_{4,3} - \xi_{4,2}^{4} + \xi_{4,1}^{10}\right)^2 \end{aligned}$$

The model algebra B of Section 6 is

$$\mathrm{B} = \mathbb{F}[\xi_{4,1},\ \xi_{4,2},\ \xi_{4,3},\ \mathbf{d}_{4,2},\ \mathbf{d}_{4,3}]/(P_1)$$

and the system of linear equations used in Lemma 6.3 looks like

$$\begin{bmatrix} \xi_{4,2} & \xi_{4,3} \\ \xi_{4,1}^{3} & -\xi_{4,2}^{3} - \xi_{4,1}^{7} \end{bmatrix} \begin{bmatrix} \mathbf{d}_{4,2} \\ \mathbf{d}_{4,3} \end{bmatrix} + \begin{bmatrix} \xi_{4,1}\mathbf{E}_4^2 \\ \xi_{4,1}\mathcal{T}_{1,2} \end{bmatrix} + \begin{bmatrix} \xi_{4,4} \\ \xi_{4,3}^{3} \end{bmatrix} = 0.$$

So, we get

$$\Delta = -\xi_{4,2}\left(\xi_{4,2}^{3} + \xi_{4,1}^{7}\right) - \xi_{4,1}^{3}\xi_{4,3}$$

and the cofactor determinant is just $\xi_{4,1}^{3}$. Finally, note that in this case the ring of invariants is a hypersurface.

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