

Corrections, Updates, and Addenda

to

Invariant Theory of Finite Groups

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Thanks for bringing typos, errors, and corrections to my attention goes to David Pengelley and Frank Williams.

§1. Corrections

I will not list punctuation mistakes or font errors unless they are too distracting.

• **PAGE 93, LINE -11**

The second half of the proof of Theorem 4.2.4 does not prove the desired statement. Starting at line -11 the proof should read as follows.

By hypothesis $p \nmid d = e_0 \cdots e_{m-1}$ so $p \nmid |(\cdots (\Sigma_{e_{m-1}} \wr \Sigma_{e_{m-2}}) \cdots \wr \Sigma_{e_1}) \wr \Sigma_{e_0}|$ and hence the restriction of the Noether map

$$\eta : \mathbb{F}[\Omega]^{(\cdots (\Sigma_{e_{m-1}} \wr \Sigma_{e_{m-2}}) \cdots \wr \Sigma_{e_1}) \wr \Sigma_{e_0}} \longrightarrow \mathbb{F}[V]^G$$

is an epimorphism. We need to show that

$$\mathbb{F}[\Omega]^{(\cdots (\Sigma_{e_{m-1}} \wr \Sigma_{e_{m-2}}) \cdots \wr \Sigma_{e_1}) \wr \Sigma_{e_0}}$$

is generated by multipolarized elementary symmetric polynomials.

This is done with an iterated application of the first fundamental theorem of invariant theory and induction on m .

For $m = 1$ this is a consequence of the first fundamental theorem for the symmetric group Σ_{e_0} , cf. page 87. So suppose $m > 1$ and we have shown that for the defining representation Ω of $(\cdots (\Sigma_{e_{m-2}} \wr \Sigma_{e_{m-3}}) \cdots \wr \Sigma_{e_1}) \wr \Sigma_{e_0}$ and any $k \in \mathbb{N}$ the algebra $\mathbb{F}[\sqcup_k \Omega]^{(\cdots (\Sigma_{e_{m-2}} \wr \Sigma_{e_{m-3}}) \cdots \wr \Sigma_{e_1}) \wr \Sigma_{e_0}}$ is generated by multipolarized elementary symmetric polynomials. Note that

$$\mathbb{F}[\Omega]^{(\cdots (\Sigma_{e_{m-1}} \wr \Sigma_{e_{m-2}}) \cdots \wr \Sigma_{e_1}) \wr \Sigma_{e_0}} = \left(\mathbb{F}[\Omega]^{(\times_{e_0} (\cdots (\Sigma_{e_{m-1}} \wr \Sigma_{e_{m-2}}) \cdots \wr \Sigma_{e_2}) \wr \Sigma_{e_1})} \right)^{\Sigma_{e_0}}.$$

Let $c = d/e_0$. If Γ is the natural permutation representation for Σ_c , then

$$\Omega = \Gamma \sqcup_{\longleftarrow e_0} \cdots \sqcup_{\longrightarrow k} \Gamma$$

as Σ_c -set. Therefore

$$\mathbb{F}[\Omega]^{(\times_{e_0} (\cdots (\Sigma_{e_{m-1}} \wr \Sigma_{e_{m-2}}) \cdots \wr \Sigma_{e_2}) \wr \Sigma_{e_1})} = \otimes_{e_0} \mathbb{F}[\sqcup_k \Gamma]^{(\cdots (\Sigma_{e_{m-1}} \wr \Sigma_{e_{m-2}}) \cdots \wr \Sigma_{e_2}) \wr \Sigma_{e_1}}.$$

Γ regarded as a $(\cdots (\Sigma_{e_{m-1}} \wr \Sigma_{e_{m-2}}) \cdots \wr \Sigma_{e_2}) \wr \Sigma_{e_1}$ -set is the defining representation so $\sqcup_k \Gamma$ is a $(\cdots (\Sigma_{e_{m-1}} \wr \Sigma_{e_{m-2}}) \cdots \wr \Sigma_{e_2}) \wr \Sigma_{e_1}$ -set to which our induction hypothesis can be applied. We therefore conclude that

$$\mathbb{F}[\sqcup_k \Gamma]^{(\cdots (\Sigma_{e_{m-1}} \wr \Sigma_{e_{m-2}}) \cdots \wr \Sigma_{e_2}) \wr \Sigma_{e_1}}$$

is generated by multipolarized elementary symmetric polynomials. Set

$$A = \mathbb{F}[\sqcup_k \Gamma]^{(\cdots (\Sigma_{e_{m-1}} \wr \Sigma_{e_{m-2}}) \cdots \wr \Sigma_{e_2}) \wr \Sigma_{e_1}}.$$

Then

$$\mathbb{F}[\Omega] \left(\times_{e_0} (\dots (\Sigma_{e_{m-1}} \wr \Sigma_{e_{m-2}}) \dots \wr \Sigma_{e_2}) \wr \Sigma_{e_1} \right) = \otimes_{e_0} \mathbb{F}[\bigsqcup_k \Gamma] \left(\dots (\Sigma_{e_{m-1}} \wr \Sigma_{e_{m-2}}) \dots \wr \Sigma_{e_2} \right) \wr \Sigma_{e_1} = \otimes_{e_0} A$$

and the action of Σ_{e_0} on $\otimes_{e_0} A$ is by permutation of the factors. Therefore by Theorem 4.1.2 and the inductive hypothesis we obtain

$$\begin{aligned} \mathbb{F}[\Omega] \left(\dots (\Sigma_{e_{m-1}} \wr \Sigma_{e_{m-2}}) \dots \wr \Sigma_{e_1} \right) \wr \Sigma_{e_0} &= (\mathbb{F}[\Omega] \left(\times_{e_0} (\dots (\Sigma_{e_{m-1}} \wr \Sigma_{e_{m-2}}) \dots \wr \Sigma_{e_2}) \wr \Sigma_{e_1} \right)) \wr \Sigma_{e_0} \\ &= (\otimes_{e_0} A)^{\Sigma_{e_0}} \end{aligned}$$

is generated by multipolarized elementary symmetric polynomials. Hence $\mathbb{F}[V]^G$ is generated by their images under the Noether map η , i.e., by fine orbit Chern classes. $\textcircled{6}$

• **PAGE 146, LINE -3**
stabilizer

• **PAGE 193, LINE -4**
Replace Vp' by $V_{p'}$.

• **PAGE 195, LINE 6**
The second part of the proof of Theorem 7.2.2 is wrong. It can be corrected as follows.

THEOREM 7.2.2: *Let $\rho : G \hookrightarrow \text{GL}(n, \mathbb{F})$ be a representation of a finite group G over the field \mathbb{F} . Assume that $\mathbb{F}[V]^G = \mathbb{F}[f_1, \dots, f_n]$ is a polynomial algebra. Then the fundamental class, $[\mathbb{F}[V]_G]$, of the ring of coinvariants is a \det^{-1} -relative invariant.*

PROOF: If G contains no transvections, then the given proof goes through.

Denote the fundamental class of $\mathbb{F}[V]_G$ by $[F]$, and assume that G contains a transvection t . Then t generates a cyclic group $\langle t \rangle$ of order p . Moreover, by the characterization of the fundamental class in Section 5.4 we have that

$$t[F] = \lambda_t [F]$$

for some $\lambda_t \in \mathbb{F}^\times$. Hence

$$\varphi : \langle t \rangle \longrightarrow \mathbb{F}^\times, \quad t \longmapsto \lambda_t$$

is a group homomorphism. Since $\langle t \rangle$ has order p we find that λ_t is a p -th root of unity and hence 1. $\textcircled{6}$

• **PAGE 200, LINE 10**
Replace \mathbb{H} by H .

• **PAGE 200, LINE 11**
Replace S set by Set .

• **PAGE 200, LINE 11**
Add the assumption that $a_H = a_{H'}$ whenever $s(H) = H'$.

• **PAGE 200, LINE 10**
Add the conclusion $s(L) = L$.

• **PAGE 200, LINE -7/6**
 $L_G = \prod_{H \in \mathcal{H}(G)} \ell_{s_H}^{|s_H| - 1}$

• **PAGE 200, LINE -4**
Note that Lemma 7.2.5 is applicable because if $s(H_{s'}) = H_{s''}$ then $s'' = ss's^{-1}$ and hence the two pseudoreflections s' and s'' have the same order.

- PAGE 231, LINE -9

The reference [4] is incorrect. The correct one is indeed missing from the reference list. So here it is:

J. Adem, *The Relations on Steenrod Powers of Cohomology Classes*, in: *Algebraic Geometry and Topology*, a symposium in honor of S. Lefschetz, pp 191-238, Princeton University Press, Princeton NJ 1957.

- PAGE 253, LINE 2

$\text{hom} - \text{codim}_{\mathbb{F}_q[V]}(M) \geq r$

- PAGE 258, LINE -4

The correct formula in the statement of Corollary 8.6.2 is

$$P^{q^k}(\mathbf{d}_{n,i}) = \begin{cases} \mathbf{d}_{n,i-1} & \text{for } k = i - 1 \geq 0, \\ -\mathbf{d}_{n,i} \mathbf{d}_{n,n-1} & \text{for } k = n - 1 \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

- PAGE 265, LINE 6

The given proof of Corollary 9.1.7 relies on Lemma 9.1.6, and hence works only for Noetherian algebras. However, direct computation shows the desired statement for all unstable algebras:

Let $f \in \sqrt{I}$. We need to show that $P^i(f) \in \sqrt{I}$ for all $i \in \mathbb{N}_0$. Since $f \in \sqrt{I}$ there exists an $s \in \mathbb{N}_0$ such that $f^{q^s} \in I$. Therefore the Cartan formulae show that

$$(P^i(f))^{q^s} = P^{iq^s}(f^{q^s}) \in I,$$

because I is \mathcal{P}^* -invariant. Hence $P^i(f) \in \sqrt{I}$. \textcircled{S}

- PAGE 283, LINE 14

Zarati

- PAGE 288, LINE -8

relevance

- PAGE 292, LINE -13

Replace \mathbf{D} by $\mathbf{D}(n)^{q^t}$ in the proof of Corollary 10.2.3 (four times).

- PAGE 293, LINE 3

$n = \dim(\mathbf{T}_U(\mathbf{D}(n)))$

- PAGE 293, LINE -3

$\varphi : H'' \rightarrow H'$

- PAGE 295, LINE 4

$\mathbf{T}_U(H') =$

- PAGE 295, LINE 14

$H'' \xrightarrow{\varphi} H'$

- PAGE 295, LINE -11

The subscript) should be erased.

- PAGE 297, LINE -18

finite dimensional

- PAGE 298ff, LINE -4

In Section 10.4 sub- and superscripts of Tor have to be switched, so that the subscript carries

the ring and the superscript the degree. This is just to make it consistent with the notation elsewhere in the book.

- **PAGE 302, LINE -2**
 $\mathbb{F}_q[z_1, \dots, z_n]$
- **PAGE 305, LINE 8**
annihilates
- **PAGE 306, LINE 12**
Lannes's
- **PAGE 306, LINE 23**
Erase line 23.
- **PAGE 307, LINE -14**
the Auslander-Buchsbaum equality
- **PAGE 310, LINE -8**
 $T_{U,\alpha}(\mathrm{Tr}^G)$:
- **PAGE 339, REFERENCE 161**
Die Frage der...
- **PAGE 340, REFERENCE 181**
S. Iyengar and L. Smith,
- **PAGE 343, REFERENCE 225**
P. S. Landweber, L. Smith, and R. E. Stong,
- **PAGE 344, REFERENCE 246**
Lannes's T-Functor and Noetherian Finiteness
- **PAGE 352, REFERENCE 413**
L. Smith and R. E. Stong
- **PAGE 353, REFERENCE 437**
R. M. W. Wood
- **PAGE 353, REFERENCE 438**
R. M. W. Wood
- **PAGE 359, COLUMN 1**
Erase lines 13 - 17.
- **PAGE 360, COLUMN 1**
Erase lines 11 - 12.
- **PAGE 360, COLUMN 1, LINE 13**
pseudoreflections
- **PAGE 364, COLUMN 1, LINE -8**
composite functor theorem
- **PAGE 364, COLUMN 2, LINE -15**
degree theorem
- **PAGE 364, COLUMN 2, LINE -6**
derivation lemma

- **PAGE 366, COLUMN 1, LINE -14**
Noetherian
- **PAGE 366, COLUMN 1, LINE -10**
first fundamental theorem
- **PAGE 366, COLUMN 1, LINE -9**
first main theorem of invariant theory
- **PAGE 368, COLUMN 2, LINE 12**
Noether normalization theorem
- **PAGE 368, COLUMN 2, LINE 20**
Noetherian finiteness
- **PAGE 370, COLUMN 1, LINE -10**
formula
- **PAGE 370, COLUMN 1, LINE -9**
subgroups
- **PAGE 371, COLUMN 1, LINE -1**
 $W(\mathbb{F}_4)$
- **PAGE 371, COLUMN 2, LINE 1**
Erase line 1.

§2. Updates on the Publication Status of Preprints in the References

- **REFERENCE 115**
M. Feshbach, *The Mod 2 Cohomology Rings of the Symmetric Groups and Invariants*, Topology 41 (2002), 57-84.
- **REFERENCE 198**
G. Kemper, *The Depth of Invariant Rings and Cohomology*, with an appendix by K. Magaard, J. of Algebra 245 (2001), 463-531.
- **REFERENCE 384**
L. Smith, *Invariants and Coinvariants of Pseudoreflection Groups, Jacobian Determinants and Steenrod Operations*, Proc. Edinb. Math. Soc. 44 (2001), 597-611.

§3. Mathematical Addenda

- **ON DEGREE BOUNDS**
New results on degree bounds, in particular on refinements of Noether's and Göbel's bounds can be found in the following papers.
- M. Sezer, *Sharpening the Generalized Noether Bound in the Invariant Theory of Finite Groups*, J. of Algebra (to appear).
- M. Sezer, *Degree Bounds for Invariants of certain Permutation Groups*, preprint, Purdue University 2002.