elements of a quotient group

$H \leq G$

$H \trianglelefteq G \iff gH = Hg \quad \forall g \in G$

$\iff gHg^{-1} \subseteq H \quad \forall g \in G$

$\iff gHg^{-1} \subseteq H \quad \forall h \in H, g \in G$

$gH := \{gh \mid h \in H\} \subseteq G$

$S = [G : H]$ = no. of distinct cosets of H in G

$G = g_1H \cup g_2H \cup \cdots \cdots \cup g_sH$ (disjoint union)

If $H \triangleleft G$ we can define an operation on the cosets

$$(g_1 H) * (g_2 H) := g_1 g_2 H$$

Using this operation the set of cosets

$$G/H = \{g_1 H, g_2 H, g_3 H, \dots, g_s H\}$$

is a group called the quotient group

<u>2.4</u> (4) The quotient group S_4/A_4 is cyclic and therefore isomorphic to $\mathbb{Z}_n$ for some $n \in \mathbb{Z}$. find n.

cyclic group of order n $G = \langle a \rangle$ such that $a^n = e$

Let's say we have another cyclic group G_2 of order n. and

$G_2 = \langle b \rangle$ with $b^n = e$

<u>proposition</u>: $G_1 \cong G_2$

Isomorphism → homomorphism

→ bijective (1-1 and onto)

$\Psi : G_1 \to G_2$

$\Psi(a^t) = b^t$

Homomorphism? $[\varphi(g_1 g_2) = \varphi(g_1)\varphi(g_2)]$

WTS $\psi(a^{t_1} a^{t_2}) = \psi(a^{t_1})\,\psi(a^{t_2})$ $\quad t_1 t_2 \in \mathbb{Z}$

LHS: $\psi(a^{t_1} a^{t_2}) = \psi(a^{t_1+t_2}) = b^{t_1+t_2}$

RHS: $\psi(a^{t_1})\,\psi(a^{t_2}) = b^{t_1} b^{t_2} = b^{t_1+t_2}$

So $\psi(a^{t_1} a^{t_2}) = \psi(a^{t_1})\,\psi(a^{t_2})$

Injective:

[if $f(x_1) = f(x_2)$
then $x_1 = x_2$]

let $\psi(a^{t_1}) = \psi(a^{t_2})$

WTS $a^{t_1} = a^{t_2}$

$$\psi(a^{t_1}) = \psi(a^{t_2})$$

$$b^{t_1} = b^{t_2} \qquad (b^n = e)$$

$\Rightarrow t_1 \equiv t_2 \pmod{n}$ i.e. $t_1 = rn + t_2$ for some $r \in \mathbb{Z}$

$\Rightarrow a^{t_1} = a^{t_2}$ (b/c $a^n = e$)

$\Rightarrow$ injective

✓

Surjective : WTS for every element $b^t \in G_2$

$\exists\, g_1 \in G$ such that $\psi(g_1) = b^t$

$a^t \in G$ and $\psi(a^t) = b^t$

so $g_1 = a^t \in G_1$ $\Rightarrow$ surjective ✓

$\Rightarrow$ isomorphism

so $G_1 \cong G_2$ i.e. All cyclic groups of the same order are isomorphic.

$\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\} = \langle 1 \rangle$ is cyclic of order n

$(+ \pmod{n})$

2.4 Q4 $\quad \frac{S_4}{A_4} \quad |S_4| = 4! = 24$

$S_4 = \{Id, (12), (13)\ (14)\ (23)\ (24)\ (34)$
$(12)(34), (13)(24), (14)(23), (123), (124), (134),$
$(234), (132), (142), (143), (243),$
$(1234), (1243), (1324), (1342), (1423), (1432)\}$

* $A_4 = \{Id, (12)(34), (13)(24), (14)(23)$
even $\quad (123), (124), (134), (234),$
permutations $\quad (132), (142), (143), (243)\}$

$$|A_4| = \frac{|S_4|}{2} = 12$$

$$\left|\frac{S_4}{A_4}\right| = \frac{|S_4|}{|A_4|} = \frac{24}{12} = 2$$

$$\frac{S_4}{A_4} = \{ \underset{\text{even}}{\overset{Id}{A_4}},\ \underset{\text{odd}}{(12)A_4} \}$$

$$[(12)A_4]^2 = (12)A_4 * (12)A_4$$

$$= (12)(12)A_4$$

$$= Id \cdot A_4 = A_4$$

$$[(123) = (13)(12)]$$

$$\frac{G}{H} = \{eH = H, g_2H, \ldots\ldots, g_5H\}$$

$$(eH) * (g_1H) = eg_1H$$

$$= g_1H$$

$\cong \langle a \rangle$ with $a^2 = e$ $\cong \mathbb{Z}_2 = \langle 1 \rangle$ with

Ans: $\left| \dfrac{S_4}{A_4} \right| = 2$

So $\dfrac{S_4}{A_4} \cong \mathbb{Z}_2$

$|S_2| = 2! = 2$

S_n is abelian $n > 3$

23. (a) $Z(G) \triangleleft G$

$Z(G) = \{ g \in G \mid gx = xg \quad \forall x \in G \}$

$H \triangleleft G \Longleftrightarrow gHg^{-1} \subseteq H \quad \forall g \in G$

$\Longleftrightarrow gHg^{-1} \subseteq H \quad \forall g \in G,\ h \in H$

Let, $x \in Z(G)$ so $xg = gx \ \forall g \in G$

let, $g \in G$ and consider $gxg^{-1} = xgg^{-1} = xe$

$= x \in Z(G)$

so $\quad g\, Z(G)\, g^{-1} \leq Z(G)$

$\Rightarrow Z(G) \trianglelefteq G$

1st isomorphism theorem

Let $\varphi : G \to H$ be a group homomorphism.

Then $G/_{\ker\varphi} \cong \mathrm{Im}(\varphi)$

* $\mathrm{Im}(\varphi) \leq H$
* $\ker\varphi \lhd G$

φ injective $\Leftrightarrow$ $\ker\varphi = \{e\}$ and $|\ker\varphi| = 1$

$$\frac{G}{\{e\}} \cong G$$

<u>Ex:</u> let, $G = GL(n, \mathbb{R}) = \{ M \in M_n(\mathbb{R}) \mid \det(M) \neq 0 \}$

$S = SL(n, \mathbb{R}) = \{ M \in M_n(\mathbb{R}) \mid \det(M) = 1 \}$

matrix multiplication

$S \triangleleft G$

WTS $\dfrac{G}{S} \cong \mathbb{R}^* \;(\text{i.e } \mathbb{Z} - \{0\})$

want $\varphi : G \to \mathbb{R}^*$

Then show $\ker \varphi = S$ | $\mathrm{Im}(\varphi) = \mathbb{R}^*$ i.e. surjective

Then use 1^{st} isomorphism theorem

$(GL(n,\mathbb{R}), *) \rightarrow (\mathbb{R}^*, *)$

multiplication

Let, $\varphi : GL(n,\mathbb{R}) \longrightarrow \mathbb{R}^*$

$$\varphi(M) = \det(M)$$

$$\ker \varphi = \{M \in GL(n,\mathbb{R}) : \varphi(M) = 1\}$$

$$= \{M \in GL(n,\mathbb{R}) : \mathrm{Det}(M) = 1\}$$

$$= SL(n,\mathbb{R})$$

WTS

$$\mathrm{Im}\,\varphi = \mathbb{R}^*$$

$$M = \begin{pmatrix} r & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix} \quad \text{with } r \in \mathbb{R}^*$$

$$\operatorname{Det}(M) = r$$

So for any $r \in \mathbb{R}^*$ there exists $M = \begin{pmatrix} r & & \\ & 1 & \\ & & \ddots \end{pmatrix} \in GL(n, \mathbb{R})$ st $\varphi(M) = r$

So φ is surjective. and

$$\operatorname{Im}(\varphi) = \mathbb{R}^*$$

So by 1st Isomorphism Theorem

$$\frac{GL(n,\mathbb{R})}{SL(n,\mathbb{R})} \cong \mathbb{R}^*$$

φ: homomorphism ?

WTS $\quad \varphi(m_1)\,\varphi(m_2) = \varphi(m_1 m_2)$

$$\varphi(m_1 m_2) = \det(M_1 M_2)$$

$$= \det(M_1)\det(M_2)$$

$$= \varphi(M_1)\,\varphi(M_2)$$

(2.5) #9 Show $\text{Inn}(S_3) \cong S_3$

$\text{Inn} = \{\varphi_g : g \in G\}$

where $\varphi_g(x) = gxg^{-1} \quad \forall x \in G$

$S_3 = \{Id, (12), (13), (23), (123), (132)\}$

$\varphi_{Id}(x) = Id\, x\, Id = x$ Id map

$\varphi_{(12)}(x) = (12)x(12) =$

$\varphi_{(13)}(x) = (13)x(13) =$

$\varphi_{(23)}(x) = (23)x(23) =$

$$\varphi_{(123)}(x) = (123)\, x\, (123)$$

$$\varphi_{(132)}(x) = (132)\, x\, (132)$$

$$\text{Inn}(S_3) = \{ \text{Id}, \varphi_{(12)}, \varphi_{(13)}, \varphi_{(23)}, \varphi_{(123)}, \varphi_{(132)} \}$$

Are all these φ_g is distinct maps

$\boxed{\underline{\text{Thm}}: \quad \dfrac{G}{Z(G)} \cong \text{Inn}(G)}$

$\boxed{\begin{aligned} &\psi : G \to \text{Inn}(G) \\ &\psi(g) = \varphi_g \end{aligned}}$

so, $G \cong \text{Inn}(G)$ Iff $Z(G) = \{e\}$

We WTS $Z(S_3) = \{e\}$

$$(12)(13) = (132) \qquad \text{so } (12), (13) \notin Z(S_3)$$
$$\neq (13)(12) = (123)$$

$$(23)(12) = (132) \qquad \text{so } (23) \notin Z(S_3)$$
$$\neq (12)(23) = (123)$$

$$(123)(12) = (13) \qquad \text{so } (123) \notin Z(S_3)$$
$$\neq (12)(123) = (23)$$

$(132)(12) = (23)$

$\neq (12)(132) = (13)$ so $(132) \notin Z(G)$

so $Z(G) = \{e\}$

$\Rightarrow S_3 \cong Im(S_3)$

Let, $a \in \mathbb{Z}_n = \{0, \dots, n-1\}$

$$|a| = n/\gcd(a,n)$$

Let, $g_1 \in G_1$, $g_2 \in G_2 \Rightarrow (g_1, g_2) \in G_1 \times G_2$

$$|(g_1, g_2)| = \operatorname{lcm}(|g_1|, |g_2|)$$

Let $(a,b) \in \mathbb{Z}_n \times \mathbb{Z}_m$

Then, $|(a,b)| = \operatorname{lcm}(|a|, |b|)$

$$= \operatorname{lcm}\left(\frac{n}{\gcd(a,n)}, \frac{n}{\gcd(b,m)}\right)$$

(2.5) #6 Isomorphism → homomorphism
→ injective
→ surjective.

$$\psi : S_3 \to S_3$$

$$\psi(x) = x^{-1}$$

homomorphism : $\psi(x_1 \cdot x_2) = \psi(x_1)\,\psi(x_2)$

$$\Leftrightarrow (x_1 x_2)^{-1} = x_1^{-1} x_2^{-1}$$

Let, $x_1 = (12)$, $x_2 = (13)$ $\in S_3$

$x_1^{-1} = (12)$, $x_2^{-1} = (13)$

$x_1 x_2 = (12)(13) = (132)$

$(x_1 x_2)^{-1} = (123)$ $\neq$

but

$x_1^{-1} x_2^{-1} = (12)(13) = (132)$

Check

$(132)(123) = (1)(2)(3) = Id$

$(123)(132) = (1)(2)(3) = Id$

So it is not a homomorphism.