Solutions to selected problems

1 a) To find a basis for the row space of ~~A~~ a matrix we reduced the matrix to echelon form using Gaussian elimination: the nonzero rows in the echelon form will be the basis for the row space.

$$A=\begin{bmatrix}1&1&1&1&0\\2&3&-1&-1&2\\3&2&1&1&3\\3&6&-1&-1&1\end{bmatrix}\sim\begin{bmatrix}1&1&1&1&0\\0&1&-3&-3&2\\0&-1&-2&-2&3\\0&3&-4&-4&1\end{bmatrix}\sim\begin{bmatrix}1&1&1&1&0\\0&1&-3&-3&2\\0&0&-5&-5&5\\0&0&5&5&-5\end{bmatrix}$$

$$\sim\begin{bmatrix}1&1&1&1&0\\0&1&-3&-3&2\\0&0&1&1&-1\\0&0&0&0&0\end{bmatrix}\sim\begin{bmatrix}1&1&0&0&1\\0&1&0&0&-1\\0&0&1&1&-1\\0&0&0&0&0\end{bmatrix}\sim\underbrace{\begin{bmatrix}1&0&0&0&2\\0&1&0&0&-1\\0&0&1&1&-1\\0&0&0&0&0\end{bmatrix}}_{\text{Reduced row echelon form}}\quad (*)$$

A basis for the row space of A will be the first three vectors of any of the matrices above. In particular, one can take $\{(1,0,0,0,2)^T, (0,1,0,0,-1)^T, (0,0,1,1,-1)^T\}$

A basis of the column space will be the column vectors of the <u>ORIGINAL</u> matrix A corresponding to the leading 1's in the reduced row echelon form. So, in this case, a basis for the column space will be the first three columns of A:

$$\left\{\begin{pmatrix}1\\2\\3\\3\end{pmatrix},\begin{pmatrix}1\\3\\2\\6\end{pmatrix},\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}\right\}$$

To find a basis for the nullspace of A, we solve the system $A\mathbf{x} = 0$.

From the reduced row echelon form ⊛ it follows that x_4 and x_5 are free variables so we let $x_4 = s$, $x_5 = t$. Thus

$$x_1 = -2x_5 = -2t$$
$$x_2 = x_5 = t$$
$$x_3 = -x_4 + x_5 = -s + t$$
$$x_4 = s$$
$$x_5 = t$$

<u>Solution</u>: $(x_1, x_2, x_3, x_4, x_5)^T = (-2t, t, -s+t, s, t)^T$
$$= t(-2, 1, 1, 0, 1)^T$$
$$+ s(0, 0, -1, 1, 0)^T$$

<u>Basis for nullspace</u>: $\{(-2,1,1,0,1)^T, (0,0,-1,1,0)^T\}$

The rank of A = dimension of row space
= 3 (# basis elements)

the nullity of A = dimension of null space
= 2 (# basis elements).

Notice that $3+2 = 5$ = # number of columns, as expected.
(Rank-Nullity theorem).

b) RREF: $\begin{bmatrix} 1 & 0 & 0 & 6 & 2 \\ 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 1 & 1 & 3 \end{bmatrix}$ rank $= 3$

Basis of nullspace: $\left\{ \begin{pmatrix} -2 \\ -4 \\ -3 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -6 \\ 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} \right\}$ nullity $= 2$

c) RREF: $\begin{bmatrix} 1 & 0 & -2/7 & 1 \\ 0 & 1 & -3/7 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ rank $= 2$

Basis of nullspace: $\left\{ \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2/7 \\ 3/7 \\ 1 \\ 0 \end{pmatrix} \right\}$ nullity $= 2$

d) RREF: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}$ rank $= 2$

Basis of nullspace: $\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$ nullity $= 1$.

2. a) To find a basis for the span of a set of vectors in $\mathbb{R}^n$, we let A be the matrix whose rows are the given vectors and reduce that matrix to echelon form. Again, a basis for the row space of A, i.e., the non-zero rows in the echelon form, will be a basis for the span of the set of vectors.

$$\begin{bmatrix} 1 & -1 & 2 & 5 \\ 0 & -1 & 3 & 1 \\ 3 & -4 & 9 & 16 \\ 1 & 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & 5 \\ 0 & -1 & 3 & 1 \\ 0 & -1 & 3 & 1 \\ 0 & 2 & -2 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & 5 \\ 0 & -1 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & -2 & -5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 5 \\ 0 & -1 & 3 & 1 \\ 0 & 2 & -2 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & 5 \\ 0 & -1 & 3 & 1 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 13/4 \\ 0 & 1 & 0 & -13/4 \\ 0 & 0 & 1 & -3/4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \leftarrow \text{reduced row echelon form}$$

A basis for the span will be

$$\{ (1, 0, 0, 13/4)^T, (0, 1, 0, -13/4)^T, (0, 0, 1, -3/4)^T \}$$

b) RREF of the corresponding matrix: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

c) RREF of the corresponding matrix: $\begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}$

3. A set of vectors spans a vector space V if every vector in V can be expressed as a linear combination of ~~the~~ vectors in the given set.

a) Let $(x,y) \in \mathbb{R}^2$, we need to determine whether there are $\alpha_1, \alpha_2, \alpha_3$ such that

$$\alpha_1(1,2)^T + \alpha_2(2,4)^T + \alpha_3(5,10)^T = (x,y)^T$$

$$(\alpha_1+\alpha_2+5\alpha_3, 2\alpha_1+4\alpha_2+10\alpha_3)^T = (x,y)^T$$

$$\alpha_1 + \alpha_2 + 5\alpha_3 = x$$
$$2\alpha_1 + 4\alpha_2 + 10\alpha_3 = y$$

Use Gaussian elimination to solve the system:

$$\left[\begin{array}{ccc|c} 1 & 1 & 5 & x \\ 2 & 4 & 10 & y \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 5 & x \\ 0 & 0 & 0 & y-2x \end{array}\right]$$

We system is consistent only if $y = 2x$, therefore <u>not every vector in $\mathbb{R}^2$</u> is in the span of the given set of vector, so they don't span $\mathbb{R}^2$.

3 c) We look at the system that arises from

$$\alpha_1 (x) + \alpha_2 (x + x^2) + \alpha_3 (-x^2) = a x^2 + b x + c$$

$$(\alpha_1 + \alpha_2) x + (\alpha_2 - \alpha_3) x^2 = a x^2 + b x + c$$

that is

$$\begin{aligned} \alpha_2 - \alpha_3 &= a \\ \alpha_1 + \alpha_2 &= b \\ 0 &= c \end{aligned}$$

Since c must be 0, not every polynomial in P_3 is in the span of this set of vectors.

d) Likewise we consider given $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{2\times 2}$ whether there are $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ such that

$$\alpha_1 \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \alpha_4 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

that is

$$\begin{pmatrix} \alpha_1 & \alpha_1 \\ 0 & \alpha_1 \end{pmatrix} + \begin{pmatrix} \alpha_2 & 0 \\ 0 & \alpha_2 \end{pmatrix} + \begin{pmatrix} 0 & \alpha_3 \\ -\alpha_3 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \alpha_4 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

or

$$\begin{pmatrix} \alpha_1 + \alpha_2 & \alpha_1 + \alpha_3 \\ -\alpha_3 & \alpha_4 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

This yields the system:

$$\begin{aligned} \alpha_1 + \alpha_2 \qquad\quad &= a \\ \alpha_1 \qquad + \alpha_3 \quad &= b \\ - \alpha_3 \quad &= c \\ \alpha_4 &= d \end{aligned}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & a \\ 1 & 0 & 1 & 0 & b \\ 0 & 0 & -1 & 0 & c \\ 0 & 0 & 0 & 1 & d \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & a \\ 0 & -1 & 1 & 0 & b-a \\ 0 & 0 & -1 & 0 & c \\ 0 & 0 & 0 & 1 & d \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & a \\ 0 & -1 & 0 & 0 & b-a+c \\ 0 & 0 & 1 & 0 & -c \\ 0 & 0 & 0 & 1 & d \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & b+c \\ 0 & 1 & 0 & 0 & a-b-c \\ 0 & 0 & 1 & 0 & -c \\ 0 & 0 & 0 & 1 & d \end{array}\right]$$

The system has a solution for each $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times 2}$,

so the set of vectors spans $\mathbb{R}^{2\times 2}$.

4) We need to determine whether for each set of vectors, there is a nontrivial linear combination giving the zero vector of the corresponding vector space.

a) Let $c_1(1,2,4)^T + c_2(2,1,3)^T + c_3(4,-1,1)^T = (0,0,0)^T$

that is $(c_1 + 2c_2 + 4c_3,\ 2c_1 + c_2 - c_3,\ 4c_1 + 3c_2 + c_3)^T = (0,0,0)^T$

$$c_1 + 2c_2 + 4c_3 = 0$$
$$2c_1 + c_2 - c_3 = 0$$
$$4c_1 + 3c_2 + c_3 = 0$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & -1 \\ 4 & 3 & 1 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \leftarrow \text{reduced row echelon form.}$$

c_3 is a free variable, so the system has nontrivial solutions and the vectors are therefore linearly dependent.

$$c_3 = t$$
$$c_2 = -3c_3 = -3t$$
$$c_1 = 2c_3 = 2t$$

Solution: $(c_1, c_2, c_3)^T = t(2,-3,1)^T$

A non-trivial comb. can be obtained by assigning values to t.

For example, we let $t=1$, and get

$$2(1,2,4)^T - 3(2,1,3)^T + 1\cdot(4,-1,1)^T = (0,0,0)^T$$

b) $c_1(x^2-2x+3) + c_2(2x^2+x+8) + c_3(x^2+8x+7) = 0\cdot x^2 + 0\cdot x + 0$

$$(c_1+c_2+c_3)x^2 + (-2c_1+c_2+8c_3)x + 3c_1+8c_2+7c_3 = 0\cdot x^2 + 0\cdot x + 0$$

Yields the system:

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ -2c_1 + c_2 + 8c_3 &= 0 \\ 3c_1 + 8c_2 + 7c_3 &= 0 \end{aligned}$$

c). $c_1\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + c_2\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} + c_3\begin{pmatrix}2 & 3\\ 0 & 2\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$

$$\begin{pmatrix}c_1+2c_3 & c_2+3c_3\\ 0 & c_1+2c_3\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

Yields the system:

$$\begin{aligned} c_1 + 2c_3 &= 0 \\ c_2 + 3c_3 &= 0 \\ c_1 + 2c_3 &= 0 \end{aligned}$$

5 a) Since the dimension of P_3 is 3, it is sufficient to determine whether the vectors are linearly independent. For that we use the Wronskian criterion

$$W(1-x, 1+x, 1-x^2) = \begin{vmatrix} 1-x & 1+x & 1-x^2 \\ -1 & 1 & -2x \\ 0 & 0 & -2 \end{vmatrix}$$

$$= -2 \begin{vmatrix} 1-x & 1+x \\ -1 & 1 \end{vmatrix}$$

$$= -2\,(1-x+1+x) = -4 \neq 0$$

Since the Wronskian is $\neq 0$, the 3 vectors are linearly independent and thus a basis for P_3.

b) We are looking at 4 vectors in $\mathbb{R}^4$, so we may use the determinant criterion to decide whether they are l.i. in $\mathbb{R}^4$ and thus a basis.

$$\begin{vmatrix} -1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{vmatrix} = ?$$

5.c) Likewise we look at

$$\det \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 3 & -1 \end{bmatrix} = ?$$

6. We have that $U \cdot [x]_U = V \cdot [x]_V$, therefore

$[x]_U = U^{-1} \cdot V [x]_V$, so $P = U^{-1} V$.

where $U = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix}$

$$U^{-1} = \begin{bmatrix} 2 & 0 & -1 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$P = U^{-1} V = \begin{bmatrix} 1 & -1 & -2 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

For $x = v_1 - 2v_2 + v_3$, $[x]_V = (1, -2, 1)^T$

so $[x]_U = \begin{bmatrix} 1 & -1 & -2 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

That is, $x = u_1 - u_2 + 2u_3$.

8. We need to determine whether the system $Ax=b$ has a solution:

a) $\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 2 & 4 & 10 \\ 1 & 2 & 5 \end{array}\right] \sim \underbrace{\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]}_{\text{consistent}}$

$x_2 = t$
$x_1 = 5 - 2t$

There are infinitely many solutions. The system is consistent. Thus b is in the column space of A.

b) $\left[\begin{array}{ccc|c} 2 & 3 & 1 & 1 \\ 1 & 1 & 1 & 3 \\ 3 & 4 & 2 & 4 \end{array}\right] \sim \cdots \sim \overset{\text{RREF}}{\left[\begin{array}{ccc|c} 1 & 0 & 2 & 8 \\ 0 & 1 & -1 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right]}$

c) $\left[\begin{array}{ccc|c} -1 & 2 & -1 & 3 \\ -2 & 2 & 1 & 8 \\ 3 & 2 & 2 & -1 \\ -3 & 8 & 5 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} -1 & 2 & -1 & 3 \\ 0 & -2 & 3 & 2 \\ 0 & 8 & -1 & 8 \\ 0 & 2 & 8 & -9 \end{array}\right] \sim \left[\begin{array}{ccc|c} -1 & 2 & -1 & 3 \\ 0 & -2 & 3 & 2 \\ 0 & 0 & +11 & 16 \\ 0 & 0 & 11 & -7 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} -1 & 2 & -1 & 3 \\ 0 & -2 & 3 & 2 \\ 0 & 0 & 11 & 16 \\ 0 & 0 & 0 & -23 \end{array}\right]$ inconsistent!

9. Let V and W be vector spaces. A mapping

$$L : V \longrightarrow W$$

is linear if for each scalar α, and each $x, y \in V$

one has,

① $L(x+y) = L(x) + L(y)$

② $L(\alpha x) = \alpha L(x)$

a) $L(x+y) = x+y+1 \neq \underset{x+1}{L(x)} + \underset{y+1}{L(y)} = x+y+2$

① fails

$L(\alpha x) = \alpha x + 1 \neq \alpha(x+1) = \alpha L(x)$

② fails.

b) $L((x_1, x_2, x_3)^T + (y_1, y_2, y_3)^T) = L((x_1+y_1, x_2+y_2, x_3+y_3)^T)$

$= (x_1+y_1, -x_3-y_3, x_1+y_1+x_2+y_2+x_3+y_3, 0)$

$= (x_1, -x_3, x_1+x_2+x_3, 0)^T + (y_1, -y_3, y_1+y_2+y_3, 0)^T$

$= L((x_1, x_2, x_3)^T) + L((y_1, y_2, y_3)^T)$

So ① holds.

$L(\alpha(x_1, x_2, x_3)^T) = L((\alpha x_1, \alpha x_2, \alpha x_3)^T)$

$= (\alpha x_1, -\alpha x_3, \alpha x_1 + \alpha x_2 + \alpha x_3, 0)^T$

$= \alpha\,(x_1, -x_3, x_1+x_2+x_3, 0)^T$

$= \alpha\, L((x_1, x_2, x_3)^T)$

2 holds as well, so L is linear.

$L(e_1) = (1, 0, 1, 0)^T$

$L(e_2) = (0, 0, 1, 0)^T$

$L(e_3) = (0, -1, 1, 0)^T$

So the standard matrix for L is

$$A = \left[L(e_1) \,\middle|\, L(e_2) \,\middle|\, L(e_3) \right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$