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SOME RESULTS ABOUT SYMMETRIC SEMIGROUPS

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ABSTRACT. In this article we prove the equivalence of two definitions of symmetric semigroups, a necessary and sufficient condition for semigroups $(n,n+a,\dots,n+sa)$, $n,a\in\mathbb{N}$, $(n,a)=1$, $2\le s\le n-2$, to be symmetric and a sufficient condition for non-symmetry of semigroups $(4,b,c,c+1)$, $b\equiv 1$, $c\equiv 2 \pmod 4$. An upper bound is given for the largest integer which is not in a semigroup generated by k relative prime positive integers.

Key words: symmetric semigroups, monomial curves, complete intersection problem.

1. INTRODUCTION

It is well known a result of Bresinsky [1] about set-theoretic complete intersection problem for monomial curves in connection with symmetric semigroups. In his Thesis Eliahou [2] proves a theorem which is a generalization of Bresinsky's result. To illustrate his theorem Eliahou offers the following monomial curves (applications 2), 3), 4), Proposition 6, ch. II):

1) $\Gamma=\{t^4,t^5,t^6,t^7\}$

This paper is based on the results of the Diploma Thesis [4]. The authoress gratefully acknowledges her Diploma's Tutor, Prof. Dr. Mario Estrada of the Cuban Academy of Sciences, for proposing her the problems and Prof. Dr. Shalom Eliahou of the University of Geneva, for his comments on this paper.

2) $\Gamma=\{t^4,t^b,t^c,t^d\}$ where

i) $b,c,d>4$,

ii) $b\equiv 1$, $c\equiv 2$, $d\equiv 3 \pmod 4$,

iii) $2c\geq b+d$.

3) $\Gamma=\{t^n,t^{n+1},t^{n+2},t^{n+3}\}$, $n\equiv -1 \pmod 3$.

The associated semigroup of the curve of 3), i.e. $\Delta=(n,n+1,n+2,n+3)$, $n\equiv -1 \pmod 3$, is a particular case of the semigroups $\Delta_1=(n,n+a,\ldots,n+sa)$, $n,a\in\mathbb{N}$, $(n,a)=1$, for $a=1$, $s=3$. The main result in this paper is the fact that the semigroups Δ_1 are symmetric if and only if $n\equiv 2 \pmod s$ (see Proposition 4). As a consequence of this proposition the related example of Eliahou results superflous. On the other hand, the examples, 1) and 2), exhibit the greater generality of his theorem. In fact, it is obvious that the associated semigroup of the curve of 1) is not symmetric, since $\mathbb{N}\setminus\Delta=\{1,2,3\}$. Moreover the associated semigroups of the curves of type 2) include those of the form $\Delta_2=(4,b,c,c+1)$, $b\equiv 1$, $c\equiv 2 \pmod 4$ $b,c>4$, $c>b+1$, which are not symmetric provided that each generator does not belong to the semigroup generated by the others. This fact is the content of Proposition 5, section 3 in this paper, so the greater generality of Eliahou's theorem is illustrated for a wide class of cases.

In addition, the equivalence of two definitions about symmetric semigroups which appear separately in [3] and [2] and other auxiliary results are proven in section 2.

NOTE. In the sequel we are going to assume that $0\in\mathbb{N}$.

2. PRELIMINARY RESULTS

Definition I: Let Δ be a cancellative semigroup with 0 element and $\tilde{\Delta}$ its associated group. Δ is called symmetric if there exists $m\in\tilde{\Delta}$ such that $\delta\in\Delta$ if and only if $m-\delta\notin\Delta$ for all $\delta\in\tilde{\Delta}$.

Remark: We restrict the former definition to a semigroup $\Delta\subset\mathbb{N}$ such that $1\notin\Delta$ (the case $\mathbb{N}=\Delta$ is trivial) and therefore $\tilde{\Delta}\subset\mathbb{Z}$.

Definition II: A semigroup $\Delta\subset\mathbb{N}$ is said to be symmetric if

a) $\mathbb{N}\setminus\Delta$ is finite

b) $\mathrm{Card}(\mathbb{N}\setminus\Delta)=\frac{c}{2}$, where $c-1$ is the largest element of $\mathbb{N}\setminus\Delta$.

Definition I occurs more frecuently in the literature about symmetric semigroups (see [1] and [3]) but we are going also to employ definition II, appearing in Eliahou [2]. Actually we have:

Proposition 1: Definitions I and II are equivalents.

Proof.

I)$\Rightarrow$II): Let Δ be a semigroup such that I holds and let's verify that there are satisfied a) and b):

a)$\mathbb{N}\setminus\Delta$ is finite because if not there exists $n\in\mathbb{N}\setminus\Delta$ such that $n>m$, that is, $m-n<0$ and by virtue of I $m-n\in\Delta\subset\mathbb{N}$ (since $n\notin\Delta$), a contradiction.

b)m is the largest element of $\mathbb{N}\setminus\Delta$: In fact, $m-m=0\in\Delta$ thus $m\notin\Delta$ (because of I), moreover $m\in\mathbb{N}$ since $1\notin\Delta \Rightarrow m-1\in\Delta \Rightarrow m=1+r$, $r\in\Delta\subset\mathbb{N}$. On the other hand, if $n>m$ then $m-n<0$ and $m-n\notin\Delta\subset\mathbb{N}$ therefore, in accordance with I, $n\in\Delta$.

From I we also derive the fact that m is odd because if not $m=2k$, $k\in\mathbb{N}$, and we have for this $k<m$ that $k\in\Delta$ if and only if $m-k=k\notin\Delta\subset\mathbb{N}$, a contradiction. Thus $m=2k+1$, for some $k\in\mathbb{N}$.

Considering now the m+1 non-negative integers $0,1,\ldots,m$ we may form the $\frac{m+1}{2}$ couples $(0,m),(1,m-1),\ldots,(\frac{m-1}{2},\frac{m+1}{2})$ each of components k and m−k, and where one and only one of these components belongs to $\mathbb{N}\setminus\Delta$. As $n>m \Rightarrow n\in\Delta$ it has to be $card(\mathbb{N}\setminus\Delta)=\frac{m+1}{2}$.

II)⇒I) Let $n_1,\ldots,n_r\in\mathbb{N}$ be the elements non-bigger than c−1 for which $n_i\in\Delta$ if and only if $(c-1)-n_i\notin\Delta$ for all $i\in\overline{1,r}$ (because of the nature of c−1 and since $\Delta\subset\mathbb{N}$ it is obvious that all the elements bigger than c−1 fulfill it). There should be therefore c−r such $k\leq c-1$, satisfaying $k\notin\Delta$ and $(c-1)-k\notin\Delta$ (because the other alternative: $k\in\Delta$ and $(c-1)-k\in\Delta$ implies $c-1\in\Delta$, a contradiction). Thus $card(\mathbb{N}\setminus\Delta)=\frac{r}{2}+c-r=c-\frac{r}{2}=\frac{c}{2}$ (the last equality being by virtue of II). Therefore r=c and m=c−1.□

Let's note as a curiosity the fact that the relationship $\mathcal{R}$ defined in $\mathbb{Z}$ by $a\mathcal{R}b$, if and only if $b=m-a$ for all $a,b\in\mathbb{Z}$ and for a certain $m\in\mathbb{Z}$ fixed, is a symmetric relationship.

The following Lemma will be also helpful.

<u>Lemma2</u>: Let $\Delta\subseteq\mathbb{N}$, $\Delta=(d_1,\ldots,d_k)$ be a semigroup with generators $d_1,\ldots,d_k\in\mathbb{N}$. Then for all $\delta\in\Delta$, $\delta\neq0$, it holds that $\delta\in\tilde{\Delta}$ if and only if there exists $j\in\overline{1,k}$ such that $\delta-d_j\in\Delta$.

<u>Proof</u>.

(⇐) Sufficiency of the condition is evident because if $\delta\in\tilde{\Delta}$ and $\delta-d_j\in\Delta$ for some $j\in\overline{1,k}$ then $\delta=d_j+d'$ with $d'\in\Delta$, thus $\delta\in\Delta$ because it is the sum of two elements of Δ.

(⇒) Let's suppose now that $\delta\in\Delta$ and $\delta\neq0$, then $\delta=\sum_{i=1}^{k}\alpha_i d_i$ where $\alpha_i\geq0$ for all $i\in\overline{1,k}$ and $\alpha_i\geq1$ for some $i\in\overline{1,k}$. So we have $\delta-d_i=\sum_{j\neq i}\alpha_j d_j+(\alpha_i-1)d_i\in\Delta$.□

<u>Remarks</u>:

i)As for all $j\in\overline{1,k}$ we have $d_j-d_j=0\in\Delta$, Lemma 2 holds clearly for the generators of Δ.

ii)If δ is an arbitrary element of Δ, by virtue of Lemma 2, we can substract from it whenever it should be necessary generators of Δ to obtain finally a generator of Δ or else zero (the later if δ is a generator itself). Therefore, this Lemma provides us a method for determining whether an arbitrary $n\in\mathbb{N}$ belongs to Δ, without solving a linear diophantine equation.

iii)The following algorithm derived from the Lemma allows us to construct semigroup Δ:

1.- Let $d=\min_{1\le i\le k} d_i$. Put $\Delta_d=\{0,d\}$.

2.- For all $n>d$ construct the set Δ_n as follows: if there is $j\in\overline{1,k}$ such that $n-d_j\in\Delta_{n-1}$ then $\Delta_n:=\Delta_{n-1}\cup\{n\}$; otherwise $\Delta_n:=\Delta_{n-1}$.

3.- $\Delta=\bigcup_{n\ge d}\Delta_n$.

Note that if $(d_1,\dots,d_k)=1$ then there is a $c\in\mathbb{Z}$ such that all subsequent integers are in Δ (In fact, by virtue of Bezout's Theorem there are $\alpha_1,\dots,\alpha_k\in\mathbb{Z}$ such that $1=\alpha_1 d_1+\dots+\alpha_k d_k$. Let it be $I=\{1,\dots,k\}$ and $J=\{i\in I:\alpha_i<0\}$ then we can take $c:=-(\sum_{j\in J}d_j-1)(\min_{j\in J}\alpha_j)\sum_{j\in J}d_j$ $=p\sum_{j\in J}d_j\in\sum_{i=1}^{k}\mathbb{N}d_i$, where $p=-(\sum_{j\in J}d_j-1)(\min_{j\in J}\alpha_j)$, since there are $t\in\mathbb{Z}$ and r, $0\le r\le\sum_{j\in J}d_j-1$, such that (Euclidean algorithm) $n=t\sum_{j\in J}d_j+r=$ $=t\sum_{j\in J}d_j+r\sum_{i\in I}\alpha_i d_i=\sum_{j\in J}(t+r\alpha_j)d_j+r\sum_{i\in I\setminus J}\alpha_i d_i$.

If now $n\ge c$ then $t\ge p$ and $t+r\alpha_j\ge p+r\alpha_j\ge 0$ for all $j\in J$. Therefore $n\in\sum_{i=1}^{k}\mathbb{N}d_i$. □ This c is consequently an upper bound for the largest integer which is not in the semigroup generated by k relative prime positive integers. For a detailed discussion on the related "coin exchange problem of Frobenius" see [5]). So if we assume that $(d_1,\dots,d_n)=1$ it should be only necessary in ii) to consider such $n\in\overline{d,c}$.

<u>Example</u>:

Semigroup $\Delta=(6,8,10,11,13)$ is symmetric.

In fact, here $k=5$, $d_1=6$, $d_2=8$, $d_3=10$, $d_4=11$, $d_5=13$. We are going to find then the elements of Δ by means of the algorithm given in iii):

1.- $d=\min_{1\le i\le 5} d_i=6$ therefore $\Delta_6=\{0,6\}$.

2.- For $n=7$, $7-d_i\notin\Delta_{n-1}$ for all $i\in\overline{1,5}$ hence $\Delta_7=\Delta_6$.

For $n=8$, $8-d_2=0\in\Delta_{n-1}$ hence $\Delta_8=\Delta_7\cup\{8\}$.

For $n=9$, $9-d_i\notin\Delta_{n-1}$ for all $i\in\overline{1,5}$ hence $\Delta_9=\Delta_8$.

For $n=10$, $10-d_3=0\in\Delta_{n-1}$ hence $\Delta_{10}=\Delta_9$.

For $n=11$, $11-d_4=0\in\Delta_{n-1}$ hence $\Delta_{11}=\Delta_{10}\cup\{11\}$.

For n=12, $12-d_1=6\in\Delta_{n-1}$ hence $\Delta_{12}=\Delta_{11}\cup\{12\}$.

For n=13, $13-d_6=0\in\Delta_{n-1}$ hence $\Delta_{13}=\Delta_{12}\cup\{13\}$.

For n=14, $14-d_1=8\in\Delta_{n-1}$ hence $\Delta_{14}=\Delta_{13}\cup\{14\}$.

For n=15, $15-d_i\notin\Delta_{n-1}$ for all $i\in\overline{1,5}$ hence $\Delta_{15}=\Delta_{14}$.

For n=16, $16-d_1=10\in\Delta_{n-1}$ hence $\Delta_{16}=\Delta_{15}\cup\{16\}$.

For n=17, $17-d_1=11\in\Delta_{n-1}$ hence $\Delta_{17}=\Delta_{16}\cup\{17\}$.

For n=18, $18-d_1=12\in\Delta_{n-1}$ hence $\Delta_{18}=\Delta_{17}\cup\{18\}$.

For n=19, $19-d_1=13\in\Delta_{n-1}$ hence $\Delta_{19}=\Delta_{18}\cup\{19\}$.

For n=20, $20-d_1=14\in\Delta_{n-1}$ hence $\Delta_{20}=\Delta_{19}\cup\{20\}$.

For n=21, $21-d_1=17\in\Delta_{n-1}$ hence $\Delta_{21}=\Delta_{20}\cup\{21\}$.

Since there are 6 consecutive integers in Δ_{21} it holds by induction for all $n\geq22$ that $\Delta_n=\Delta_{n-1}\cup\{n\}$, since $n-d_1\in\Delta_{n-1}$.

3.-$\Delta=\cup_{n\geq6}\Delta_n=\{0,6,8,10,11,12,13,14,16\rightarrow\}$ (the final arrow indicates henceforth that all subsequent integers belong to Δ).

Therefore $\mathbb{N}\setminus\Delta=\{1,2,3,4,5,7,9,15\}$ and the largest element of $\mathbb{N}\setminus\Delta$ is m=15. On the other hand $\mathrm{card}(\mathbb{N}\setminus\Delta)=8=\frac{m+1}{2}$. That is, Δ is symmetric in accordance with definition II.

The next Lemma is indispensable for proving Proposition 4.

<u>Lemma3</u>: Let it be $\Delta=(n,n+a,\dots,n+sa)$, $n,a\in\mathbb{N}$, $(n,a)=1$, $2\leq s\leq n-2$. Then the largest element of $\mathbb{N}\setminus\Delta$ is $m=\frac{n-q}{s}n+(n-1)a$, where q is the least integer such that $q\geq2$ and $n-q\equiv0 \pmod s$.

<u>Proof</u>.

The following <u>remarks</u> are going to be helpful:

iv)If $\beta a\in\Delta$ then either $\beta=0$ or $\beta\geq n$, because if $\beta a=\sum_{i=0}^{s}\beta_i(n+ia)=\sum_{i=0}^{s}\beta_i n+\sum_{i=0}^{s}\beta_i ia$, $\beta_i\geq0$, then $(\beta-\sum_{i=0}^{s}\beta_i i)a=(\sum_{i=0}^{s}\beta_i)n$. But $(n,a)=1$, thus n divides $(\beta-\sum_{i=0}^{s}\beta_i i)\geq0$ and therefore either it holds equality to zero and consequently $\sum_{i=0}^{s}\beta_i=0$ and $\beta=\sum_{i=0}^{s}\beta_i i=0$ or else $\beta-\sum_{i=0}^{s}\beta_i i\geq n$ whence $\beta\geq n$ since $\sum_{i=0}^{s}\beta_i i\geq0$.

v)If $n+\alpha a\in\Delta$, $\alpha\in\mathbb{N}$, then in accordance with Lemma 2, there is $i\in\overline{0,s}$, such that $n+\alpha a-(n+ia)=(\alpha-i)a\in\Delta$ and by iv) $\alpha-i=0$ or $\alpha-i\geq n$, i.e., $\alpha\geq i+n$. Therefore if $n+\alpha a\in\Delta$ with $\alpha<n$, then $\alpha\in\overline{0,s}$.

vi)If $0\leq h\leq ts$ then $tn+ha\in\Delta$ because it can be expressed as the sum of t elements of Δ of the form $n+ja$, $j\in\overline{0,s}$ (i.e., generators of Δ).

vii)If $\delta\geq\delta'\in\Delta$ and $\delta\equiv\delta' \pmod n$ then there is $q\geq0$ such that $\delta=qn+\delta'$, thus $\delta\in\Delta$.

First of all $m\notin\Delta$ because if $m\in\Delta$ then by virtue of Lemma2 there is $j_1\in\overline{0,s}$ such that $m-(n+j_1a)=(\frac{n-q}{s}-1)n+(n-1-j_1)a\in\Delta$.

Applying now Lemma2 to this element we find then $j_2\in\overline{0,s}$ such that $(\frac{n-q}{s}-1)n+(n-1-j_1)a-(n+j_2a)=(\frac{n-q}{s}-2)n+(n-1-j_1-j_2)a\in\Delta$, and so forth until we find $j_{\frac{n-q}{s}-1}\in\overline{0,s}$ such that $n+(n-1-\sum_{k=1}^{\frac{n-q}{s}-1} j_k)a\in\Delta$.

However, because of remark v), $(n-1-\sum_{k=1}^{\frac{n-q}{s}-1} j_k)\in\overline{0,s}$. Thus $0\leq n-1-\sum_{k=1}^{\frac{n-q}{s}-1} j_k\leq s$, that is to say,

$$n-1-s\leq\sum_{k=1}^{\frac{n-q}{s}-1} j_k\leq(\frac{n-q}{s}-1)s \quad \text{(since } j_k\leq s \text{ for all } k\text{)}$$

and then

$$n-1-s\leq n-q-s\leq n-2-s \quad \text{(since } q\geq 2\text{)}.$$

This contradiction shows that $m\notin\Delta$.

Let now $k\in\overline{q-1,n-1}$. It holds then $n-1-k\leq\frac{n-q}{s}s=n-q$, so in accordance with observation vi) $\frac{n-q}{s}n+(n-1-k)a\in\Delta$.

On the other hand, if $r\in\overline{1,q-1}$ then $\frac{n-q}{s}n+(n-q+r)a+n=$ $=\frac{n-q}{s}n+(n-q)a+n+ra=\frac{n-q}{s}(n+sa)+n+ra\in\Delta$, since $r\leq q-1\leq s$ (for the way we choosed q: if $n\equiv t$ (t=the remainder mod s) (mod s) and $t\geq 2$ then it is clear that $q=t<s$, otherwise $q=s$ or else $q=s+1$ and always $q-1\leq s$).

In short, we have included in Δ the set of n remainders (mod n) $A=A_1\cup A_2$, where

$A_1=\{\frac{n-q}{s}n+(n-1-k)a:\ k\in\overline{q-1,n-1}\}$ and

$A_2=\{\frac{n-q}{s}n+(n-q+r)a+n:\ r\in\overline{1,q-1}\}$.

Let $h\in\overline{1,n}$ and let $m+h$ be the n's integers subsequent to m. There are then n-q of them congruent (mod n) to the elements of A_1 and the remaining q are congruent to the elements of A_2.

For those congruent to the elements of A_1 there is not doubt about their belonging to Δ. Let it now be

$$\frac{n-q}{s}n+(n-1)a+h\equiv\frac{n-q}{s}n+(n-q+r)a+n \quad \text{(mod n)}$$

then

$$\frac{n-q}{s}n+(n-1)a+h\geq\frac{n-q}{s}n+(n-q+r)a+n$$

because

$$\frac{n-q}{s}n+(n-1)a+h\equiv\frac{n-q}{s}n+(n-q+r)a \quad \text{(mod n)}$$

and the left side of the foregoing congruence is bigger than the right side. Whence, in accordance with remark vii), $\frac{n-q}{s}n+(n-1)a+h\in\Delta$, and the Lemma is proved because of remark vii) again. □

3. SOME CLASSES OF SYMMETRIC AND NON-SYMMETRIC SEMIGROUPS

Proposition4: The semigroups of the form $\Delta=(n,n+a,\dots,n+sa)$, $n,a\in\mathbb{N}$, $(n,a)=1$ and $2\le s\le n-2$ are symmetric if and only if $n\equiv 2 \pmod s$.

Proof.

($\Leftarrow$) If $n\equiv 2 \pmod s$, by Lemma 3, $m=\frac{n-2}{s}n+(n-1)a$ is the largest integer which is not in Δ.

Now we are ready to prove that Δ holds definition I with the foregoing m, i.e., for all $\delta\in\tilde{\Delta}$, $\delta\in\Delta$ if and only if $m-\delta\notin\Delta$.

Necessity of the condition holds because if not then $\delta\in\Delta$ and also $m-\delta\in\Delta$, and so $\delta+(m-\delta)=m\in\Delta$, in opposition to Lemma 3.

To prove sufficiency it is enough to regard only such δ that $m-\delta>0$ and $m-\delta\notin\Delta$ because if $m-\delta<0$ then $m<\delta$ and $\delta\in\Delta$ by the nature of m.

Let it be therefore $\delta<m$ such that $m-\delta\notin\Delta$. In particular $m-\delta\not\equiv 0 \pmod n$, thus it must be congruent (mod n) to one of the ka, $k=1,\dots,n-1$. It means that there are $q\in\mathbb{Z}$ and $k\in\overline{1,n-1}$ such that

$$\frac{n-2}{s}n+(n-1)a-\delta=qn+ka\notin\Delta \quad (1)$$

so $k>qs$ (if $q\le 0$ it is obvious, if $q>0$ and $k\le qs$ then by remark vi), Lemma 3, $qn+ka\in\Delta$, a contradiction), that is, $k\ge qs+1$; whence

$$n-2-qs\ge n-1-k \quad (2)$$

Solving in (1), $\delta=(\frac{n-2}{s}-q)n+(n-1-k)a$ and inequality (2) shows that $\delta\in\Delta$ by virtue of remark vi), ib.

($\Rightarrow$) First of all we are going to prove the following:

Assertion: Let Δ and q be as in the statement of Lemma 3, then $m=\frac{n-q}{s}n+(n-q+1)a\notin\Delta$.

In fact, as in the proof of Lemma 3, by *reductio ad absurdum*, if we assume that $m\in\Delta$ then there are $j_1,\dots,j_{\frac{n-q}{s}-1}\in\overline{0,s}$ such that

$$n+(n-q+1-\sum_{k=1}^{\frac{n-q}{s}-1} j_k)a\in\Delta$$

hence

$$0\le n-q+1-\sum_{k=1}^{\frac{n-q}{s}-1} j_k\le s \text{ and}$$

$$n-q+1-s\le \sum_{k=1}^{\frac{n-q}{s}-1} j_k\le n-q-s, \text{ a contradiction.}$$

Therefore, $m_1\notin\Delta$.

It remains only to prove that if $q>2$ then Δ is not symmetric. But it is simple because $m_1\notin\Delta$ and $m-m_1=\frac{n-q}{s}n+(n-1)a-(\frac{n-q}{s}n+(n-q+1)a)$ $=(q-2)a$, where $0<q-2<s<n$.

Thus, in accordance with remark iv) Lemma 3, $m-m_1\notin\Delta$. and Δ doesn't fulfill definition I. $\square$

Examples:

1) $\Delta=(35,43,51,59,67,75,83,91,99,107,115,123)$ is symmetric by virtue of Proposition 4, with $n=35$, $a=8$, $s=11$. We also have the largest integer which is not in Δ, $m=377$. (Observe that without the above proposition it could be annoying to study the symmetry of Δ and the computation of m as well.)

2) The associated semigroup in the example 3) of Eliahou (see Introduction), $\Delta=(n,n+1,n+2,n+3)$, $n\equiv-1 \pmod 3$, is symmetric since $-1\equiv2 \pmod 3$ and we can make use of Proposition 4, with $a=1$, $s=3$.

Proposition5: The semigroups of the form $\Delta=(4,b,c,c+1)$, $b,c>4$, $b\equiv1$, $c\equiv2 \pmod 4$ and $c\geq b+1$ are not symmetric if we require that each generator does not belong to the semigroup generated by the others.

Proof.

We shall employ definition II of symmetric semigroup and split the proof in two parts:

a) Let's assume first that $b=c-1=4r+1$, $r\in\mathbb{N}$, that is, $\Delta=(4,4r+1,4r+2,4r+3)$. Here is not possible for a generator to belong to the semigroup generated by the others. In this case Δ is never symmetric since $\Delta=\{4,8,\ldots,4r,4r+1,4r+2,4r+3\rightarrow\}$ and so, the largest integer which is not in Δ is $m=4r-1$, meanwhile $\mathbb{N}\setminus\Delta=\{1,2,3,5,\ldots,4r-3,4r-2,4r-1\}$ and $\mathrm{card}(\mathbb{N}\setminus\Delta)=3r\neq\frac{m+1}{2}=2r$, since $r\geq1$.

b) Let now $b=4r+1$, $c=4k+2$, and $r\neq k$, that is, $\Delta=(4,4r+1,4k+2,4k+3)$. It can be checked out that the requirements of our proposition holds if and only if $r<k<2r$. Then

$\Delta=\{4,8,4r,4r+1,\ldots,4k,4k+1,4k+2,4k+3\rightarrow\}$, and

$\mathbb{N}\setminus\Delta=\{1,2,3,5,6,7,\ldots,4r+2,4r+3,\ldots,4(k-1)+2,4(k-1)+3\}$.

On the other hand $m=4k-1$, and it doesn't hold definition II since $\frac{m+1}{2}=2k\neq\mathrm{card}(\mathbb{N}\setminus\Delta)=r+2k$. $\square$

Example:

$\Delta=(4,b,c,d)$, with $b,c,d>4$, $b\equiv1$, $c\equiv2$, $d\equiv3 \pmod 4$ and $2c\geq b+d$, in the second example of Eliahou presented in the introduction, for $d=c+1$, is not symmetric provided that each generator does not belong to the semigroup generated by the others.

REFERENCES

[1] BRESINSKY, H. Monomial Gorenstein curves in $\mathbb{A}^4$ as set-theoretic complete intersection. Manuscripta Math. 27 (1979) 353-358.

[2] ELIAHOU, S. Courbes monomiales et Algébre de Rees Symbolique. Thése No.2080 Université de Genéve, 1983 (French).

[3] HERZOG, J. Generators and relations of abelian semigroups and semigroups rings. Manuscripta Math. 3 (1970) 175-193.

[4] JUAN, L. Symmetric semigroups and monomial curves. Diploma Thesis. University of La Habana, 1991 (Spanish).

[5] RÔDSETH, Ô.J. On a linear Diophantine problem of Frobenius. Jo. Für Reine Angewandte Mathematik. Vol. 301 (1978) 171-178.

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