Differential ‘Galois’ extensions with new constants

Extensions différentielles « galoisiennes » avec nouvelles constantes

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\textbf{A R T I C L E I N F O}

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\textbf{A B S T R A C T}

Let $F$ be a differential field with algebraically closed field of constants $C$ and let $E$ be a differential field extension of $F$. The field $E$ is a differential Galois extension if it is generated over $F$ by a full set of solutions of a linear homogeneous differential equation with coefficients in $F$ and if its field of constants coincides with $C$. We study the differential field extensions of $F$ that satisfy the first condition but not the second.

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\textbf{R É S U M É}

Soit $F$ un corps différentiel dont le corps des constantes $C$ est algébriquement clos et soit $E \supset F$ une extension de corps différentiels. Le corps différentiel $E$ est une extension galoisienne différentielle de $F$ s’il est engendré sur $F$ par une base de solutions d’une équation différentielle linéaire homogène à coefficients dans $F$ et si son corps des constantes est $C$. Nous étudions les extensions différentielles de $F$ qui satisfont la première condition et non la seconde.

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\section{Introduction}

Let $F$ be a differential field of characteristic zero. Let $C$ denote the field of constants of $F$ (assumed algebraically closed). Let $L = Y^{(n)} + a_1 Y^{(n-1)} + \cdots + a_n Y$ be a linear differential operator over $F$.

A Picard–Vessiot, or differential Galois, extension $E$ of $F$ for $L$ is a differential field extension, also with field of constants $C$ (i.e. a no new constant extension), generated over $F$ as a differential field by $n$ solutions of $L = 0$ linearly independent over $C$ (i.e. a full set of solutions). The differential Galois group $G(E/F)$ is the group of differential field automorphisms of $E$ fixing $F$, and the resulting Galois correspondence includes the fact that the fixed field of $E$ under $G(E/F)$ is $F$. The same extension $E$ is a Picard–Vessiot extension of $F$ for many different operators $L$. One way to suppress the explicit reference to a specific operator is to note that a differential field extension $E \supset F$ is a Picard–Vessiot extension for some operator $L$ provided that

1. There is a group $G$ of differential automorphisms of $E$ over $F$ whose fixed field is $F$.
2. $E$ is generated over $F$ as a differential field by a $G$-submodule $V$ which is finite dimensional over $C$.
3. $E$ is a no new constant extension of $F$.

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The operator $L$ for which $E$ is Picard–Vessiot is produced from $V$ using Wronskian determinants, and then $V$ has a full set of solutions of $L$ as a $C$-basis (see [2]).

There are natural situations, however, for example the case of certain derivations on coordinate rings of connected algebraic groups, which lead to extensions $E \supseteq F$ which meet the first two criteria but not the third. Such extensions are our subject here.

We re-formulate the situation in terms of rings generated by solutions.

The ring $F[y_{ij} : 1 \leq i \leq n, 0 \leq j \leq n-1][w^{-1}]$, where $w = \det(y_{ij})$, with derivation $y'_{ij} = y_{i,j+1}$, $y'_{i,0} = -a_{n-1}y_{i,n-1} - \cdots - a_0y_{i,0}$ contains and is differentially generated by $n$ solutions $y_{i,0}$ to the differential equation $L = 0$ whose Wronskian is a unit, and the same remains true for its homomorphic images. (This is why it is called a full universal solution algebra for $L$ in [2].) If we take a maximal differential ideal of the full universal solution algebra and mod it out, the quotient field of the resulting integral domain is a Picard–Vessiot extension of $F$. When the prime ideal is not maximal, the constant field $K$ of $E$ is a proper extension of $F$, and of course need not be algebraically closed. Our main result shows that nonetheless $E$ is much like a Picard–Vessiot extension of the compositum field $FK$. In particular, we find an algebraic subgroup $G$ of $GL_n(C)$ with $E^G = FK$.

Unless (0) is a maximal differential ideal of the full universal solution algebra, the latter will always have non-maximal prime differential ideals to which our results apply. A typical situation to which our results apply is the following: Let $H$ be a connected algebraic group over $C$, let $D_1, \ldots, D_m$ be a basis of $\text{Lie}(H)$, let $b_1 \in F$ and consider the derivation $D = D_F \otimes 1 + \sum b_i \otimes D_i$ on $F[H] = F \otimes C[H]$. We show that there is an operator $L$ as above which has a full set of solutions in $F(H)$ (the quotient field of $F[H]$) with invertible Wronskian, so that the algebra they differentially generate over $F$ fits into our theory above. The appropriate consequences are drawn.

We will preserve throughout the notational conventions of this introduction. Another useful reference on the Picard–Vessiot theory is [3].

2. Modulo non-maximal primes

Let $P$ be a prime differential ideal of the full universal solution algebra $F[y_{ij} : 1 \leq i \leq n, 0 \leq j \leq n-1][w^{-1}]$ for $L = Y^{(n)} + a_1 Y^{(n-1)} + \cdots + a_0 Y$, let $R_F$ denote $F[y_{ij}][w^{-1}]$, and let $E$ denote the quotient field of $R_F / P$. Let $K$ denote the field of constants of $E$. We choose an algebraic closure $\overline{E}$ of $E$. Then $\overline{K}$, the algebraic closure of $K$ in $\overline{E}$, is also algebraically closed and is the field of constants of the compositum $\overline{E} \overline{K}$ and $F \overline{K}$. To see that $\overline{K}$ is indeed the field of constants, we use the fact that $\overline{E} \overline{K} \supseteq \overline{E}$ is an algebraic extension, so that any constant must be algebraic over the constants $K$ of $E$.

**Lemma 1.** $\overline{E} \overline{K}$ is a Picard–Vessiot extension of $F \overline{K}$ for $L$.

**Proof.** $\overline{E} \overline{K}$ is generated over $F \overline{K}$ by a full set of solutions, with non-zero Wronskian of $L = 0$, so the only issue is new constants. As we just noted, the constants of $\overline{E} \overline{K}$ are $\overline{K}$ which is also the constants of $F \overline{K}$. □

Let $E_0$ be a differential subfield of $E$ containing $F$ and $K$, for example $FK$. Then Lemma 1 also implies that $\overline{E} \overline{K}$ is a Picard–Vessiot extension of $E_0 \overline{K}$.

Let $z_{ij}$ be the image of $y_{ij}$ in $\overline{E} \overline{K}$. Let $R_{E_0}\overline{K} = E_0 \overline{K} \otimes_F R_F$ and consider the homomorphism $R_{E_0}\overline{K} \to \overline{E} \overline{K}$ induced from $y_{ij} \mapsto z_{ij}$. Let $M$ be its kernel. Note that $R_{E_0}\overline{K}$ is also a full universal solution algebra for $L$ over $E_0 \overline{K}$. If $M$ were not maximal then as above the quotient field $E_{0}\overline{K} / M$ would contain a new constant, a contradiction. So $M$ is a maximal differential ideal.

Let $R_{E_0} = E_0 \otimes_F R_F$ and let $M_0$ be the kernel of the homomorphism $R_{E_0} \to E$ induced from $y_{ij} \mapsto z_{ij}$. We have $R_{E_0}\overline{K} \supseteq R_{E_0}$ and $M \cap R_{E_0} = M_0$.

We note that $\overline{K} \cap E = K$, since anything algebraic over constants is a constant. It follows that $\overline{K} \otimes_K E$ is an integral domain, and hence so is $\overline{K} \otimes_K E_0$. Since these are integral domains algebraic over fields, they are themselves fields. Thus $\overline{E} = \overline{K} \otimes_K E$ and $E_0 \overline{K} = \overline{K} \otimes_K E_0$.

**Lemma 2.** $E_0 \overline{K} \cap E = E_0$.

**Proof.** From the above equalities, the assertion is the obvious one that $(\overline{K} \otimes_K E_0) \cap (K \otimes_K E) = K \otimes_K E_0$ in $\overline{K} \otimes_K E$. □

These observations allow us to conclude that the ideal $M$ is induced (we use the above notation):
Lemma 3. $\bar{K} \otimes_K R_{E_0}/M_0$ is isomorphic to $R_{E_0}/M$ as $\bar{K}$-algebra. In particular:

1. $M_0$ is a maximal differential ideal.
2. $M = R_{E_0}\bar{K}M_0$ as ideals.
3. $M = \bar{K}M_0$ as $\bar{K}$-vector spaces.

Proof. By the discussion preceding the lemma, the subring $S = \bar{K} \otimes_K R_{E_0}/M_0$ of $\bar{K} \otimes_K E$ can be regarded as a subring of $ER$. Viewed in that way, it is the $E_0\bar{K}$ algebra generated by $z_{ij}$ and the inverse Wronskian, and that algebra is $R_{E_0}\bar{K}/M$. This implies the main assertion of the lemma and the others are direct consequences. □

The rings $R_{E_0}$ and $R_{E_0}\bar{K}$ are generated by the vector spaces $V_K = \sum_{i,j} K y_{ij}$ and $V_{\bar{K}} = \sum_{i,j} \bar{K} y_{ij}$ (and the inverse Wronskians) over their coefficient fields. Differential actions of the groups $GL_n(K)$ and $GL_n(\bar{K})$ on $R_{F_0}$ and $R_{F_0}\bar{K}$ come from their actions on these vector spaces [2]. The inclusion $GL_n(K) \subseteq GL_n(\bar{K})$ is compatible with the inclusion $V_K \subseteq V_{\bar{K}}$. So we can regard $GL_n(K)$ as acting on $R_{E_0}\bar{K}$, and the restriction of that action to the subring $R_{E_0}$ is the given action. The stabilizer $GL_n(\bar{K})_M$ of the ideal $M$ is an algebraic subgroup of $GL_n(\bar{K})$; in fact, it is the differential Galois group $G(E\bar{K}/E_0\bar{K})$ [2].

Lemma 4. The stabilizer $GL_n(K)_{M_0}$ of $M_0$ is Zariski dense in $GL_n(\bar{K})_M$.

Proof. We have by Lemma 3 that $M = \bar{K}M_0$. If $M_0$ were a finite dimensional $K$-vector space, the result would be obvious. Since the actions here are rational, we can reduce immediately to the finite dimensional case and conclude the same result. □

We are now ready for the main result.

Theorem 1. Let $G(E/E_0) = GL_n(K)_{M_0}$. Then $E^{G(E/E_0)} = E_0$.

Proof. Because $G(E/E_0)$ is Zariski dense in $G(E\bar{K}/E_0\bar{K})$ by Lemma 4, and $E\bar{K} \supset E_0\bar{K}$ is Picard–Vessiot by Lemma 1 we have $(E\bar{K})^{G(E/E_0)} = E_0\bar{K}$. It follows that $E^{G(E/E_0)} \subseteq E_0\bar{K}$. So it suffices to show that $E \cap E_0\bar{K} = E_0$. This is just Lemma 2. □

3. Derivations of group coordinate rings

Warning: the symbol $E$ is used in this section differently than above.

Let $H$ be a connected linear algebraic group over $C$, let $D_1, \ldots, D_m$ be a basis of Lie($H$), let $b_1 \in F$ and consider the derivation $D = D_1 \otimes 1 + 1 \otimes D_1$ on $F[H] = F \otimes C[H]$. We regard $H$ as acting on $F[H]$ by left translations $(h \cdot f(g) = f(gh))$ so that $D$ is $H$-equivariant. Both $D$ and the $H$-action extend to the quotient field $F(H)$ and commute with each other.

Let $W$ be some finite dimensional $C$-subspace which is $H$-stable and generates $C[H]$ as a $C$-algebra. Then the following properties hold for the differential field extension $E = F(H) \supset F$:

1. The group $H$ is a group of differential automorphisms of $E$ over $F$ such that $E^H = F$.
2. There is a finite dimensional $H$-stable, $C$-vector space $W \subset E$ such that $E = F(W)$ is differentially generated over $F$ by $W$.

As remarked in the introduction, if additionally the constants of $E$ were those of $F$, $E$ would be a Picard–Vessiot extension of $F$. We do not make that assumption here; hence any field extension meeting the above two criteria is called a pre-Picard–Vessiot (briefly pPV) extension of $F$. For example, if $X$ is an irreducible $H$-torsor and $D$ a derivation on $F(X)$ induced by an element of the corresponding twisted Lie algebra (see [1] for details) then $F(X)$ is a pPV extension of $F$.

Let $K$ be the field of constants of an arbitrary pPV extension $E$. Let $V = KW$, where $W$ is as defined above except that we can replace $C[H]$ by the coordinate ring $C[X]$ of an irreducible $H$-torsor, and let $z_1, \ldots, z_m$ be a $K$-basis of $W$. Note that the Wronskian $w(z_1, \ldots, z_m)$ is non-zero by construction. $H$ acts on $KW$, although not $K$-linearly in general. Nonetheless, for $h \in H$ we have a matrix $\alpha(h) \in GL_n(K)$ such that

$$(z_1^h, \ldots, z_m^h) = (z_1, \ldots, z_m)\alpha(h).$$

We can differentiate both sides of this equation and obtain

$$(z_1)^h, \ldots, (z_m)^h = (z_1', \ldots, z_m')\alpha(h)$$

using the fact that $K$ is the field of constants. Repeated differentiation shows that a similar formula holds for higher derivatives as well.
Now suppose $Y$ is a differential indeterminate over $F$, and consider the Wronskian determinant $w(Y, z_1, \ldots, z_n)$ in $E[Y]$. When we expand this determinant along the first column, the various minors that occur have rows of the form $(z_1^{(i)}, \ldots, z_m^{(i)})$. These rows transform under $h \in H$ as above via multiplication by $\alpha(h)$. This applies to the coefficient $w(z_1, \ldots, z_m)$ of $Y^{(m)}$ as well. It then follows that the coefficients of $L = w(z_1, \ldots, z_m)^{-1}w(Y, z_1, \ldots, z_n) = Y^{(m)} + a_1Y^{(m-1)} + \cdots + a_m Y^{(0)}$ are invariant under any $h \in H$ and hence lie in $F$. (This is an adaptation of the argument in [2].)

We thus have a homomorphism from the full universal solution algebra $R_F = F[y_{ij}][w^{-1}]$ over $F$ to $E$ by $y_{ij} \mapsto z_i$ whose kernel is a prime ideal $P$. Let $E_1$ denote the quotient field of its image. By construction, $F$ is a subfield of $E_1$, and $E = E_1K$. The constants of $E_1$ are $K_1 = K \cap E_1$. By Theorem 1, we have $FK_1 = E_1^{G(E_1/FK_1)}$, where $G(E_1/FK_1)$ is a subgroup of $\text{GL}_m(K_1)$. Thus the extension $E \supset F$ breaks into the subextensions $E = E_1K \supset E_1 \supset E_1^{G(E_1/FK_1)} = FK_1 \supset F$, where the extensions on the ends are by constants and that in the middle is by a group.

References