PICARD–VESSIOT EXTENSIONS WITH SPECIFIED GALOIS GROUP

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ABSTRACT. Picard-Vessiot extensions are determined by their differential module structure. For a fixed group G Picard-Vessiot extensions with differential Galois group G are all isomorphic as G-modules but not as differential rings. It is then shown that isomorphism classes of Picard-Vessiot extensions with group G correspond to G-orbits in a certain finite dimensional vector space with G-action.

1. INTRODUCTION

The normal basis theorem states that if K/L is a finite Galois extension of fields with group Γ, then K is isomorphic to the group ring $L[\Gamma]$ as an $L[\Gamma]$-module. In this paper we will consider a counterpart of this result in differential Galois theory. This counterpart leads to considering the problem of recognizing a Picard-Vessiot extension E of a differential field F from weaker information than the structure of E as a differential field. We will first recall from [3] and [2] some basic definitions and then state our main results.

Let F be a differential field of characteristic 0 having an algebraically closed field of constants C. A field extension E/F is a Picard Vessiot extension if E is differentially generated over F by a full set of solutions of a monic homogeneous differential equation over F and if E has the same field of constants as F (see [3], [2]). The Picard Vessiot ring R of E/F may be described as the ring of elements of E whose iterated images under the derivation D_E of E span a finite dimensional vector space over F. The fraction field of R is equal to E. The differential Galois group G of E/F is the affine algebraic group over C given by the differential automorphisms of E over F. Let $C[G]$ be the affine coordinate ring of G over C, and let $F[G] = F \otimes_C C[G]$.

Our counterpart of the normal basis Theorem is the following result.

Theorem 1. *The Picard Vessiot ring R is isomorphic to $F[G]$ as an $F[G]$-comodule.*

The normal basis theorem implies that the structure of K as an $L[\Gamma]$-module is not sufficient in general to determine K as a Galois extension of

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F. Similarly, Theorem 1 shows that the $F[G]$-comodule structure of R does not in general determine R or E. It is a natural question to try to minimize, in various ways, the amount of information about E which is sufficient to determine E. Our first result along this line is proved in Proposition 1 of §2:

Theorem 2. *The structure of E as a differential F-module is sufficient to determine E.*

In general E has infinite dimension as an F-vector space. To try to determine E from a finite amount of linear algebra data over F, one could simply note that the differential equation over F which determines E is specified by finitely many coefficients in F. More conceptually, the differential equation can be replaced by an associated finite dimensional differential module over F which is trivialized by E (see [3]). However, there is no preferred choice for a differential equation giving E or for an associated differential module. We will consider the following more canonical method. This method depends on choosing a faithful finite dimensional G-module V which admits an embedding into $C[G]$.

We show in Proposition 2 of §2 that there is an isomorphism $\alpha : R \to F[G]$ respecting F and G-module structure. Using this isomorphism, we can transport the derivation D_R of R to a G-endomorphism of $F[G]$ which makes $F[G]$ into a differential F-module. Define $\mathcal{W}$ to be the sum of the images of V in $F[G]$ under G-homomorphisms. We will show in Lemma 5 of §4 that $\mathcal{W}$ is a finite dimensional F-vector space. We will also prove that $\mathcal{W}$ is a rational G-module which is stable under the above differential structure coming from D_R, and that this restriction determines D_R. Let Δ_R be the resulting differential structure on $\mathcal{W}$. The construction of $\mathcal{W}$ and Δ_R is canonical up to the choice of the F and G-module isomorphism $\alpha : R \to F[G]$. Let $\mathrm{End}_{F.G}(\mathcal{W})$ (resp. $\mathrm{Aut}_{F.G}(\mathcal{W})$) be the group of endomorphisms (resp. automorphisms) of $\mathcal{W}$ which respect the actions of F and G on $\mathcal{W}$. As a corollary of Theorem 1 of §4 we will prove the following result:

Theorem 3. *Isomorphism classes of Picard-Vessiot extensions E of F with differential Galois group G correspond to the orbits of $\mathrm{Aut}_{F.G}(\mathcal{W})$ on $\mathrm{End}_{F.G}(\mathcal{W})$.*

We note that $\mathcal{W}$ is a two-sided G-module. This observation permits the F and G endomorphism ring of $\mathcal{W}$ to be calculated in principle. We include a number of examples in §5.

We retain throughout the notations of this introduction. In addition, we refer to F-vector spaces with rational G action trivial on F as $F \cdot G$-modules. Some basic results on $F \cdot G$-modules are collected in §3 below. We define an $F \cdot D$ module to be an F-vector space with a derivation which (i) is compatible with the derivation of F, and (ii) for which the derivation iterates of each element form a finite dimesional F-vector space. The usual group algebra of G over F will be denoted $F\langle G\rangle$.

2. Differential Modules

We begin by showing that the differential F module structure alone distinguishes Picard–Vessiot extensions.

Proposition 1. *Let E_i, $i = 1, 2$ be Picard–Vessiot extensions of F, and assume that they are isomorphic as differential F vector spaces. Then they are isomorphic as differential fields. This is equivalent to the Picard–Vessiot rings of the E_i being isomorphic as differential F vector spaces.*

Proof. We may assume that both E_1 and E_2 are differential subfields of a Picard–Vessiot extension of F. Let $B : E_1 \to E_2$ be an F linear differential isomorphism. Let $R_i \subset E_i$ be the set of elements which satisfy monic linear homogeneous differential equations over F. Suppose $\alpha \in E_1$ is in R_1 and satisfies the differential equation $L(\alpha) = 0$ where $L(X) = X^{(n)} + a_{n-1}X^{(n-1)} + \cdots + a_0$ is a monic homogeneous linear differential operator over F. We may assume that $L^{-1}(0) \subset E_1$ and $\dim_C L^{-1}(0) = n$. For any $\beta \in L^{-1}(0)$, $0 = B(L(\beta)) = L(B(\beta))$, so $B(L^{-1}(0)) = L^{-1}(0)$ and $\alpha \in L^{-1}(0) = B(L^{-1}(0)) \subset R_2$. It follows that $R_1 \subseteq R_2$. Similar considerations apply to elements of R_2. We conclude both that $R_1 = R_2$, and also that the isomorphism B carries R_1 to R_2. Since R_i is the Picard-Vessiot ring of E_i and E_i is the quotient field of R_i, we have that $E_1 = E_2$. Thus the identity is the desired differential field isomorphism between E_1 and E_2; note that it does not, in general, coincide with B. An examination of the proof shows that it suffices to begin with an $F \cdot D$ isomorphism from R_1 to R_2. □

Proposition 1 implies that it is sufficient to consider the $F \cdot D$ module (R, D_R), that is, the F vector space R with its designated endomorphism D_F (which is C, but not necessarily F, linear). We now consider the $F \cdot G$-module structure of R.

Proposition 2. *Let R be the Picard–Vessiot ring of a Picard–Vessiot extension E of F with differential Galois group G. Then R is isomorphic to $F[G]$ as an $F \cdot G$ module.*

Proof. It is a consequence of Kolchin's Theorem [2, Theorem 5.12] that there is a finite Galois extension F_1 of F such that $F_1 \otimes_F R \cong F_1[G]$ as F_1 and G modules. Let $n = [F_1 : F]$. As F vector spaces with G action, the two sides of the above are isomorphic to $R^{(n)}$ and $F[G]^{(n)}$, which means the two direct sums are isomorphic as $F \cdot G$ modules. This implies that the socles of the direct sums are isomorphic $F \cdot G$ modules, and then, by counting multiplicities of simple components, that the socles of R and $F[G]$ are $F \cdot G$ isomorphic. Since $F[G]$ is $F \cdot G$ injective [1], the isomorphism of the direct sums implies that R is also an injective $F \cdot G$ module. Finally, injective $F \cdot G$ modules with isomorphic socles are isomorphic. □

3. $F \cdot G$ modules

We recall that $F[G] = F \otimes_C C[G]$. We associate F valued functions on G to elements of $F[G]$: if $f = \sum a_i \otimes \phi_i$ then for $g \in G$ $f(g) = \sum \phi_i(g)a_i$. It is clear that elements of $F[G]$ are determined by their associated functions. Using this functional representation, the actions of G on $F[G]$ are as follows.

The left action: $\lambda(g)f = g \cdot f$ where $g \cdot f(x) = f(xg)$.

The right action: $\rho(g)f = f \cdot g^{-1}$ where $f \cdot g(x) = f(gx)$.

Note that $\lambda(g)\rho(h) = \rho(h)\lambda(g)$.

We have the evaluation functionals given by $\text{ev}_g : F[G] \to F$ by $\text{ev}_g(f) = f(g)$. Note that 0 is the only element of $F[G]$ in the kernel of all the evaluation functionals.

If $X \subset F[G]$ is a left (or right) $F \cdot G$ submodule, we use λ (or ρ) to denote the G action on X, and we use $\text{ev}_g : X \to F$ for the evaluation restrictions. In $\text{End}_F(X)$ we let $\Lambda = \sum_g F\lambda(g)$ (or $P = \sum_g F\rho(g)$). If X is both a left and right $F \cdot G$ submodule then Λ and P commute.

Suppose X is finite dimensional. Then the fact that intersections of the kernels of the evaluation functionals is trivial means that they span $\text{Hom}_F(X, F)$.

That $F[G]$ is $F \cdot G$ injective, and that the multiplicities of the simple components of its socle are finite follow from the following lemma, whose proof and use are the same as the familiar case where $F = C$ [1, Prop. 1.4, p. 9]:

Lemma 1. *Let W be a finite dimensional $F \cdot G$ module. Let $\gamma_W : W \to W \otimes_F F[G]$ be the associated $F[G]$ comodule structure, and let $ev_e : F[G] \to F$ be evaluation at the identity. Then there is an isomorphism as (right) G modules and F vector spaces*

$$Hom_{F \cdot G}(W, F[G]) \to Hom_F(W, F)$$

given by $\Phi \mapsto ev_e \circ \Phi$ and $f \mapsto (1 \otimes f) \circ \gamma_W$.

This "duality" lemma also implies the following structural result about endomorphism rings:

Lemma 2. *Let $W \subset F[G]$ be a finite dimensional $F \cdot G$ submodule which is also a right G submodule, and suppose every $F \cdot G$ morphism $W \to F[G]$ has image in W. Then*

$$End_{F \cdot G}(W) = P.$$

Proof. By assumption we have

$$\text{End}_{F \cdot G}(W) = \text{Hom}_{F \cdot G}(W, F[G])$$

and Lemma 1 says that

$$\text{Hom}_{F \cdot G}(W, F[G]) \to \text{Hom}_F(W, F)$$

given by $\Phi \mapsto \text{ev}_e \circ \Phi$ is an isomorphism. For $g \in G$ we have that $\rho(g^{-1})$ is an $F \cdot G$ endomorphism of W, and one checks that $\text{ev}_e \circ \rho(g^{-1}) = \text{ev}_g$. Since W

is finite dimensional, $\mathrm{Hom}_F(W, F)$ is spanned by evaluations, which means that the subring P of the $F \cdot G$ endomorphism ring maps onto $\mathrm{Hom}_F(W, F)$ and hence coincides with the endomorphism ring. □

Another way to state the result of Lemma 2 is that the ring homomorphism

$$F\langle G\rangle \to \mathrm{End}_{F\cdot G}(W)$$

induced from

$$\rho : G \to \mathrm{End}_{F\cdot G}(W)$$

is surjective.

This is not necessarily the case for automorphisms. However, we have the following:

Lemma 3. *Let W be an $F \cdot G$ submodule of $F[G]$. The morphisms*

$$End_{F\cdot G}(F[G]) \to End_{F\cdot G}(\mathcal{W})$$

and

$$Aut_{F\cdot G}(F[G]) \to Aut_{F\cdot G}(\mathcal{W})$$

induced from restriction are surjective.

Proof. By construction, $\mathcal{W}$ is an $F \cdot G$ submodule of the injective $F \cdot G$ module $F[G]$. This means that $F[G]$ contains an injective hull I of $\mathcal{W}$. Then I is an essential extension of $\mathcal{W}$, which in our context means that they have the same socle, and I, being injective is also a direct summand of $F[G]$. Let B be an automorphism of $\mathcal{W}$. Composing B with the inclusion of $\mathcal{W}$ in I is a monomorphism. Since I is injective, the inclusion of $\mathcal{W}$ into I factors through this monomorphism, so that B lifts to an endomorphism B_0 of I. The same argument, in the case that B is only an endomorphism, also proves the first assertion of the lemma. The kernel of B_0, if non-trivial, contains a simple submodule, which belongs to the socle of I and therefore the socle of $\mathcal{W}$. This simple module then is contained in $\mathcal{W}$, and hence in the kernel of B. That kernel is trivial, and thus so is the kernel of B_0. The image $I_0 = B_0(I)$ is then an injective submodule of I (being isomorphic to I) and contains $B(\mathcal{W}) = \mathcal{W}$. Again, because I is an essential extension of $\mathcal{W}$ this implies that $I = I_0$ and B_0 is onto, and hence an automorphism of I_0. Now take B_1 to be an automorphism of $F[G]$ which is B_0 on I and the identity on a complementary direct summand. □

There is a derivation of $F[G]$ coming from F, given by $D_F \otimes 1$. We denote this by ∂ in this section. Note that ∂ is a G morphism. Suppose that X is a finite dimensional $F \cdot G$ submodule of $F[G]$ with $\partial(X) \subset X$ and that $T : X \to X$ is an $F \cdot G$ endomorphism. We define:

$$T' = \partial \circ T - T \circ \partial.$$

It is straightforward to check that T' is also an $F \cdot G$ endomorphism. If $T = 1 \otimes \tau$ for τ a G endomorphism of $C[G]$ then T commutes with ∂ and

$T' = 0$. It follows that

$$\text{if } T = \sum f_i \rho(g_i) \text{ then } T' = \sum f_i' \rho(g_i).$$

which we call the *differentiation of coefficients formula*. In these notations, we also have the following conjugation formula:

Lemma 4. *Let X be a finite dimensional $F \cdot G$ submodule of $F[G]$ with $\partial(X) \subset X$ and $B : X \to X$ an $F \cdot G$ automorphism. Then*

$$B^{-1} \partial B = \partial + B^{-1} B'.$$

More generally, if (X, Δ) is an $F \cdot D$ structure on X and $T = \Delta - \partial$ then $B^{-1} \Delta B = \partial + B^{-1} B' + B^{-1} T B$.

Proof. Let $S = B^{-1} \partial B$, so $BS = \partial \circ B$. By definition, $\partial \circ B = B' + B \circ \partial$, so $S = B^{-1}(B' + B \circ \partial) = B^{-1} B' + \partial$, as desired. The second formula is immediate from the first. □

4. Isomorphism classes of Picard-Vessiot extensions

Let V be a faithful finite dimensional G-module over C. We will need the following definition.

Definition 1. *Let Y be any rational G module over C. Then $\mathcal{W}(Y) = \sum \{\phi(V) | \phi \in Hom_G(V, Y)\}$. For $Y = F[G]$, we let $\mathcal{W}$ denote $\mathcal{W}(F[G])$.*

If $f : Y \to Z$ is a G module morphism, then $f(\mathcal{W}(Y)) \subseteq \mathcal{W}(Z)$, and in particular $\mathcal{W}(Y)$ is stable under G endomorphisms of Y. It is clear that $\mathcal{W}(Y)$ is a G submodule of Y, and that $\mathcal{W}(Y)$ is the image of

$$\mathrm{Hom}_G(V, Y) \otimes_C V \to Y$$

by

$$\phi \otimes y \mapsto \phi(y).$$

In the special case $Y = F[G]$ and $W = F \otimes_C V$ we have

$$\mathrm{Hom}_G(V, F[G]) = \mathrm{Hom}_{F \cdot G}(W, F[G])$$

and

$$\otimes_C V = \otimes_F W$$

so that

$$\mathrm{Hom}_G(V, F[G]) \otimes_C V = \mathrm{Hom}_{F \cdot G}(W, F[G]) \otimes_F W$$

from which it follows that $\mathcal{W}(F[G])$ is the image of

$$\mathrm{Hom}_{F \cdot G}(W, F[G]) \otimes_F W \to F[G]$$

by

$$\psi \otimes w \mapsto \psi(w).$$

Since $\mathrm{Hom}_{F \cdot G}(W, F[G]) = \mathrm{Hom}_F(W, F)$ is a finite dimensional F module, this shows that $\mathcal{W}(F[G])$ is a finite dimensional $F \cdot G$ module, and that $\mathcal{W}(F[G]) = \sum \{\psi(W) | \psi \in \mathrm{Hom}_{F \cdot G}(W, F[G])\}$. Since R is $F \cdot G$ isomorphic to $F[G]$, we see that $\mathcal{W}(R)$ is also a finite dimensional $F \cdot G$ submodule of R. Moreover, the restriction of D_E to $\mathcal{W}(R)$ determines D_E, as we now note:

Lemma 5. *$\mathcal{W}(R)$ is finite dimensional over F and an $F \cdot G$ and $F \cdot D$ submodule of R. The F subalgebra of R generated by $\mathcal{W}(R)$ has quotient field E. In particular, the restriction of D_E to $\mathcal{W}(R)$ determines D_E.*

Proof. D is a G endomorphism of R, and hence preserves $\mathcal{W}(R)$. This makes $\mathcal{W}(R)$ an $F \cdot D$ submodule of R, and hence the subalgebra generated over F by $\mathcal{W}(R)$ is a differential subalgebra of R, and its quotient field K is then an intermediate differential field of the Picard–Vessiot extension $E \supset F$, so of the form E^H for a subgroup H of G. By assumption, we have an embedding $V \to C[G]$, hence $V \to F[G]$, and therefore, by Proposition 2, an embedding $\phi : V \to R$. Then $\phi(V)$ is a faithful G submodule of $\mathcal{W}(R)$ and hence of K. Since no element of G other then e acts trivially on K, we have H trivial and $K = E$. □

We are now ready to construct the invariant. We recall from Proposition 1 that if we have two Picard–Vessiot rings R_i, $i = 1, 2$ with corresponding derivations D_i, then they are isomorphic as Picard–Vessiot rings if and only if there is an F vector space isomorphism $B : R_1 \to R_2$ such that $D_2 = BD_1B^{-1}$. If there is such a B, then, there will be one which is an $F \cdot G$ isomorphism.

We can select an $F \cdot G$ isomorphism $A_i : R_i \to F[G]$, as per Proposition 2 and consider the G endomorphisms $A_iDA_i^{-1}$ of $F[G]$.

We have $A_i(\mathcal{W}(R_i)) = \mathcal{W}$, and hence, by Lemma 5, that $\Delta_i = A_iDA_i^{-1}$ is determined by its restriction to $\mathcal{W}$, which we denote by the same symbol. As previously noted, the structures $\mathcal{M}_i = (\mathcal{W}, \Delta_i)$ are $F \cdot D$ modules. If the R_i are isomorphic, then clearly so are the $\mathcal{M}_i$, where by the latter we mean that there is an $F \cdot G$ module automorphism of $\mathcal{W}$ carrying Δ_1 to Δ_2. We record this, and its converse in the following result:

Theorem 1. *Let E_i, $i = 1, 2$ be Picard–Vessiot extensions of F with group G. Let R_i be the Picard–Vessiot ring of E_i and $A_i : R_i \to F[G]$ an $F \cdot G$ isomorphism. Then the E_i are isomorphic if and only if there is an $F \cdot G$ automorphism B of $\mathcal{W}$ such that*

$$BA_1D_{R_1}A_1^{-1}B^{-1}|_{\mathcal{W}} = A_2D_{R_2}A_2^{-1}|_{\mathcal{W}}$$

Proof. If the E_i are isomorphic, then there is a differential $F \cdot G$ isomorphism $R_1 \to R_2$ which produces B. Conversely, suppose we have B. If B is the restriction to $\mathcal{W}$ of an $F \cdot G$ automorphism B_1 of $F[G]$, then replacing A_1 by B_1A_1 gives an isomorphism of R_1 to $F[G]$ such that the resulting $F \cdot D$ module structure on $\mathcal{W}$ coincides with that for R_2, so that the R_i and hence E_i are isomorphic. So the theorem follows from Lemma 3. □

Theorem 1 says that an isomorphism class of Picard–Vessiot extensions corresponds to an equivalence class of differential structures on $\mathcal{W}$, the equivalence relation coming from conjugation by $F \cdot G$ automorphisms. We always have the differential structure ∂ on $\mathcal{W}$ induced from $D_F \otimes 1$ on $F[G]$ as in

§3. Then $(\mathcal{W}, \Delta)$ is an $F \cdot D$ module if and only if $\Delta - \partial$ is an $F \cdot G$ endomorphism of $\mathcal{W}$. The action of $F \cdot G$ automorphisms on differential structures on $\mathcal{W}$ then translates to the following action on $F \cdot G$ endomorphisms:

For $B \in \mathrm{Aut}_{F.G}(\mathcal{W})$ and $T \in \mathrm{End}_{F.G}(\mathcal{W})$ let $T^B = BTB^{-1} + B^{-1}B'$, where $B' = \partial \circ B - B \circ \partial$. This defines a right action of automorphisms on endomorphisms, called, for obvious reasons, conjugation plus logarithmic differentiation. Theorem 1 and Lemma 4 imply the following:

Corollary 1. *Isomorphism classes of Picard–Vessiot extensions of F with group G correspond to $Aut_{F.G}(\mathcal{W})$ orbits on $End_{F.G}(\mathcal{W})$ under the conjugation plus logarithmic differentiation right action.*

In the next section we will compute some examples of this action and some Picard–Vessiot extensions.

5. Examples

Throughout this section we use $\mathrm{dlog}(x)$ to denote the logarithmic derivative $\frac{x'}{x}$.

We begin with the case of finite G. Finite Galois extensions $E \supset F$ are Picard–Vessiot: there is a unique extension of the derivation D_F to E, and this turns out to have field of constants C. Moreover, in this case the Picard–Vessiot ring R coincides with E. On the other hand, it is never the case that R is isomorphic as an F algebra with $F[G]$, as the latter is always just a finite product of copies of F.

Example 1. $G = \mathbb{Z}/2\mathbb{Z}$

A Picard–Vessiot extension of F with group G is then of the form $E = F(\sqrt{d})$ where d is a non-square of F. Assume D_F is extended to E. Differentiating the equation $(\sqrt{d})^2 = d$ shows that $(\sqrt{d})' = \frac{d'}{2\sqrt{d}}$ which we write as $\frac{1}{2}\mathrm{dlog}(d)\sqrt{d}$

Let e denote the identity and g denote the non-trivial element of G. The ring $F[G]$ is all functions F^G, which is isomorphic as a G module to the group algebra $F\langle G\rangle = Fe + Fg$. (The isomorphism has e corresponding to the constant function 1 and g to the function which is 1 on the identity and -1 on g.) There is a G isomorphism $A : E \to F\langle G\rangle$ by $1 \mapsto e + g$ and $\sqrt{d} \mapsto e - g$. (This is, of course, a special case of the Normal Basis Theorem.) In terms of coordinates, $A(a + b\sqrt{d}) = (a + b)e + (a - b)g$ while $A^{-1}(\alpha e + \beta g) = \frac{\alpha+\beta}{2} + \frac{\alpha-\beta}{2}\sqrt{d}$. Then if $D = D_E$ we calculate

$$ADA^{-1}(\alpha e + \beta g) = (\alpha' + \frac{\alpha - \beta}{2}\frac{1}{2}\mathrm{dlog}(d))e + (\beta' - \frac{\alpha - \beta}{2}\frac{1}{2}\mathrm{dlog}(d))g.$$

The derivation $\partial = D_F \otimes 1$ of $F\langle G\rangle$ is given by $\partial(\alpha e + \beta g) = \alpha' e + \beta' g$ (see §3) and hence the determining $F \cdot G$ module endomorphism $T = ADA^{-1} - \partial$ is given by

$$T(\alpha e + \beta g) = \frac{1}{2}\mathrm{dlog}(d)(\frac{\alpha - \beta}{2}e + \frac{\beta - \alpha}{2}g).$$

At this point we specify the faithful G module V: we choose the one dimensional module on which g acts non-trivially, which appears here as the module spanned by $e-g$. It then follows that we may choose $F(e-g)$ for $\mathcal{W}$. On $\mathcal{W}$, T becomes multiplication by $\frac{1}{2}\mathrm{dlog}(d)$. Now suppose B is any $F\cdot G$ automorphism of $\mathcal{W}$. Then B is multiplication by some nonzero element α of F, so that B commutes with T and $B^{-1}B'$ is multiplication by $\mathrm{dlog}(\alpha)$ and $B^{-1}TB+B^{-1}B'$ is multiplication by $\frac{1}{2}\mathrm{dlog}(d)+\mathrm{dlog}(\alpha)$ which can be written $\frac{1}{2}\mathrm{dlog}(\alpha^2 d)$.

This is interpreted as follows: Picard–Vessiot extensions of F with group G are quadratic extensions of F. Those isomorphic to $F(\sqrt{d})$ are of the form $F(\sqrt{c})$ where c is equivalent to d modulo squares, or $c=\alpha^2 d$ for some $\alpha\in F$.

Example 2. $G=\mathbb{G}_m$

For Picard–Vessiot extensions $E\supset F$ with group $\mathbb{G}_m$, we have that $E=F(y)$ where y is transcendental over F and satisfies $y'=ay$ for some $a\in F$. Furthermore, there is no $\alpha\in F$ with $\alpha'=a\alpha$, which says that a is not a logarithmic derivative in F. The Picard–Vessiot ring here is $F[y,y^{-1}]$, which is isomorphic as an F algebra and G module to $F[\mathbb{G}_m]$, the isomorphism A carrying y to the coordinate t on $\mathbb{G}_m$. For V we can take the C module spanned by t, and then $\mathcal{W}$ turns out to be Ft. We compute ADA^{-1} on $\mathcal{W}$: $\alpha t\mapsto \alpha y\mapsto(\alpha'+a)y\mapsto(\alpha'+a)t$. So $T=ADA^{-1}-\partial$ is multiplication by a. As in Example 1, an $F\cdot G$ automorphism B of $\mathcal{W}$ is multiplication by some non–zero element b of F, and so $B^{-1}TB+B^{-1}B'$ is multiplication by $a+\mathrm{dlog}(b)$.

Suppose K is a Picard–Vessiot extension of F with group $\mathbb{G}_m$ which is $\mathbb{G}_m$ differentially isomorphic to $C(y)$. Modeling E on $C(y)$ we see that E is generated by $z=by$ for some non–zero $b\in F$. Then z satisfies $z'=(b'+ab)y=(\mathrm{dlog}(b)+a)z$. Thus the invariant corresponding to K is $a+\mathrm{dlog}(b)$.

Example 3. $G=\mathrm{SL}_2(C)$

As with Example 2, for $G=\mathrm{SL}_2(C)$ all Picard–Vessiot rings are isomorphic to $F[\mathrm{SL}_2(C)]$ (see [2, Theorem 5.12] and [4, Proposition 33]). However, in this case the classification of extensions is not available, and so we confine our attention to the specific case $F=C(x)$, the field of rational functions with constant coefficients and with $x'=1$, and the Picard–Vessiot extension $E\supset F$ for the Airy equation $Y''-xY$. Then the Picard–Vessiot ring R is known to be $F[y,z,y',z']/(yz'-zy'-1)$, where y,z are solutions of the Airy equation [2, Example 4.29].

In terms of the familiar matrix coordinates we can write

$$F[\mathrm{SL}_2]=F[x_{11},x_{12},x_{21},x_{22}]/(x_{11}x_{22}-x_{12}x_{21}-1).$$

There is an obvious F algebra isomorphism $A:R\to F[\mathrm{SL}_2]$ by $y\mapsto x_{11}$, $z\mapsto x_{21}$, $y'\mapsto x_{12}$, and $z'\mapsto x_{22}$.

For V, we are going to use the SL_2 module C^2 (column 2-tuples with the usual left matrix multiplication action of SL_2). V appears in R as $Cy + Cz$, which we will use. In order for A to be SL_2 linear, we need to use the *right* action of SL_2 on $F[\mathrm{SL}_2]$: thus if X is the matrix $[x_{ij}]$ and $g \in \mathrm{SL}_2(C)$ is the matrix $[x_{ij}(g)]$ then $x_{ij}^g = x_{ij}(gX) = x_{i1}(g)x_{1j} + x_{i2}(g)x_{2j}$.

One checks then that $\mathcal{W}$ is 4 dimensional over F, and hence equals $\sum Fx_{ij}$. Then $T = ADA^{-1}$ is given by $T(x_{i1}) = x_{i2}$ and $T(x_{i2}) = xx_{i1}$.

To determine the class of T, we need to know about the $F \cdot G$ automorphisms of $\mathcal{W}$. According to Lemma 2, all $F \cdot G$ endomorphisms of $\mathcal{W}$ are F linear combinations of "right" (here left) SL_2 translations symbolized by $X \mapsto Xg$. Every 2 by 2 matrix P in $M_2(F)$ can be written as an F linear combination of matrices in $\mathrm{SL}_2(C)$. Thus every $F \cdot G$ endomorphism B of $\mathcal{W}$ can be written as $X \mapsto XP$ for some P in $M_2(F)$. For example, T is represented by the matrix $\begin{bmatrix} 0 & x \\ 1 & 0 \end{bmatrix}$. An endomorphism B will be an automorphism if and only if the representing matrix P is invertible, in which case B^{-1} will be represented by P^{-1}. By the derivation of coefficients formula below, we know that B' is given by the matrix obtained from P by differentiating entries.

So let B be an $F \cdot G$ automorphism of $\mathcal{W}$ represented by an invertible matrix P which we write in the form δQ where Q has determinant 1.

Let $Q = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Then $B^{-1}TB + B^{-1}B'$ is represented by the matrix

$$\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} 0 & x \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} + \mathrm{dlog}(\delta)$$

References

[1] E. Cline, B. Parshall, and L. Scott, Induced modules and affine quotients, Math. Ann. **230** (1977), 1–14.

[2] A. Magid, Lectures on Differential Galois Theory, University Lecture Series **7**, American Mathematical Society, Providence RI, 1997 (second printing with corrections).

[3] M. van der Put, M. F. Singer, Differential Galois Theory, Springer-Verlag, 2003.

[4] J.-P. Serre, Galois Cohomology, Springer-Verlag, 1997.

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