The Differential Azumaya Algebras and Non-commutative Picard–Vessiot Cocycles

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Abstract

A differential Azumaya algebra, and in particular a differential matrix algebra, over a differential field K with constants C is trivialized by a Picard–Vessiot (differential Galois) extension E. This yields a bijection between isomorphism classes of differential algebras and Picard–Vessiot cocycles $Z^1(G(E/K), PGL_n(C))$ which cobound in $Z^1(G(E/K), PGL_n(E))$.

1 Introduction

Let K be a differential field with algebraically closed characteristic zero field of constants C and derivation $D = D_K$. By a *differential Azumaya algebra* over K we mean a pair $(A, \mathcal{D})$ where A is an Azumaya (central separable) K algebra and $\mathcal{D}$ is a derivation of A extending the derivation D of its center K. Given two such pairs $(A_1, \mathcal{D}_1)$ and $(A_2, \mathcal{D}_2)$, we can form the differential Azumaya algebra $(A_1 \otimes A_2, \mathcal{D}_1 \otimes \mathcal{D}_2)$, where the derivation $\mathcal{D}_1 \otimes \mathcal{D}_2$ is given by $\mathcal{D}_1 \otimes 1 + 1 \otimes \mathcal{D}_2$. Morphisms of differential Azumaya algebras over K are K algebra homomorphisms preserving derivations. We denote the isomorphism class of $(A, \mathcal{D})$ by $[A, \mathcal{D}]$. The above tensor product defines a monoid operation on the isomorphism classes (the identity being the class $[K, D_K]$), and hence we can form the corresponding universal group $K_0^{\text{diff}}\mathcal{A}z(K)$. The corresponding object for K when derivations are not considered is the familiar $K_0\mathcal{A}z(K)$ as defined by Bass in [1, Chapter III], and there is an obvious group homomorphism $K_0^{\text{diff}}\mathcal{A}z(K) \to K_0\mathcal{A}z(K)$ given by $[A, \mathcal{D}] \mapsto [A]$. This group homomorphism is surjective: any derivation, such as D_K, of the center of any Azumaya K

algebra A can be extended to a derivation of A (Hochschild's Theorem; see [5]). These extensions can be done in many ways (we recall, for later use, that any two differ by a derivation trivial on the center, and hence inner [1, 1.6, p. 86]). For the matrix algebras $M_n(K)$, we have the extension which is given by D_K on the matrix coordinates, and we denote the corresponding differential Azumaya algebra $(M_n(K), (\quad)')$. We note that under the usual isomorphism from $M_p(K) \otimes M_q(K) \to M_{pq}(K)$ we have $(M_p(K), (\quad)') \otimes (M_q(K), (\quad)')$ isomorphic to $(M_{pq}(K), (\quad)')$.

The group homomorphism $K_0\mathcal{A}z(K) \to Br(K)$ from the universal isomorphism class group of K to its Brauer group [1, 6.2, p. 113] is the surjection with kernel the subgroup $K_0\mathcal{M}at(K)$ generated by the matrix algebras. We consider the corresponding subgroup M^1 of $K_0^{\text{diff}}\mathcal{M}at(K$ generated by the algebras $(M_n(K), (\quad)')$ and its quotient $\mathcal{B}r^{\text{diff}}(K)$. We call this quotient the *differential Brauer Group* of K. It will be of interest even in the case that K is algebraically closed (when $Br(K)$ is trivial, in which case $\mathcal{B}r^{\text{diff}}(K)$ reduces to M^2/M^1, where the group M^2 is generated by the differential Azumaya algebras $[M_n(K), \mathcal{D}]$ whose K algebra component is a matrix algebra.).

Of course such a K still usually has a substantial differential Galois theory, with connected Galois groups. In fact, for any given differential Azumaya algebra $(A, \mathcal{D})$, we may make a finite Galois (hence also uniquely differential Galois) base extension E of K such that with this extension of scalars A becomes a matrix algebra and we have $(A, \mathcal{D}) \otimes E$ in M^2 for $K_0^{\text{dif}}\mathcal{A}z(K)$. Thus, for the relations with the connected differential Galois theory, we can even concentrate on the group M^2/M^1 for our base field, which we do below. Once this group is analyzed we return to the general situation in our final section. In this context, the algebras $(M_n(K), (\quad)')$ play the role of matrix algebras in the non–differential theory, and hence we will refer to them as the *trivial* differential Azumaya algebras.

We retain throughout the paper the terminology, notation, and conventions of this introduction.

2 Differential Matrix Algebras

We begin with an analysis of the structure of a differential Azumaya algebra whose underlying algebra is a matrix algebra.

Definition 1. *Let P be an $n \times n$ matrix over K. Then $\mathcal{D}_P$ denotes the derivation of $M_n(K)$ given by*

$$\mathcal{D}_P(X) = (X)' + PX - XP.$$

Note that $\mathcal{D}_P = \mathcal{D}_Q$ implies that $P - Q$ is central, and hence scalar, so $P = Q + aI_n$ for suitable $a \in K$, and conversely. And we note that

$$(M_n(K), \mathcal{D}_P) \otimes (M_m(K), \mathcal{D}_Q) = (M_{nm}(K), \mathcal{D}_{P \otimes I_m + I_n \otimes Q}).$$

It is a simple consequence of the fact that central derivations of Azumaya algebras are inner to see that all matrix differential Azumaya algebras have such derivations:

Proposition 1. *Let $(M_n(K), \mathcal{D})$ be a differential Azumaya K algebra. Then there is a matrix $P \in M_n(K)$ such that $\mathcal{D} = \mathcal{D}_P$.*

Proof. $\mathcal{D} - (\quad)'$ is a central derivation of $M_n(K)$, and hence inner. If this inner derivation is given by $P \in M_n(K)$, then $\mathcal{D}(X) - (X)' = PX - XP$, so that $\mathcal{D} = \mathcal{D}_P$. □

From now on, we will denote a matrix differential Azumaya algebra as $(M_n(K), \mathcal{D}_P)$, where it is understood that P is only defined up to addition of scalar multiples of the identity.

Using this structure, we consider when a matrix differential Azumaya algebra is trivial.

Proposition 2. *$(M_n(K), \mathcal{D}_P)$ is isomorphic to $(M_n(K), \mathcal{D}_Q)$ if and only if there is $H \in GL_n(K)$ and $b \in K$ such that $H^{-1}H' + H^{-1}QH = P + bI_n$. In particular, $(M_n(K), \mathcal{D}_P)$ is isomorphic to $(M_n(K), (\quad)')$ if and only if there is $F \in GL_n(K)$ and $a \in K$ such that $F^{-1}(F)' = P + aI_n$.*

Proof. Suppose $T : (M_n(K), \mathcal{D}_P) \to (M_n(K), \mathcal{D}_Q)$ is a differential K algebra isomorphism. As a K algebra isomorphism (actually automorphism) of $M_n(K)$, T is inner [6, Thm. 2.10, p.16]: there is $H \in GL_n(K)$ such that $T(X) = HXH^{-1}$. Note that H is only defined up to a non–zero scalar multiple. Since T is differential, we have $T(\mathcal{D}_P(X)) = \mathcal{D}_Q(T(X))$, so

$$(HXH^{-1})' + QHXH^{-1} - HXH^{-1}Q = H(X' + PX - XP)H^{-1}.$$

We expand: $(HXH^{-1})' = H'XH^{-1} + HX'H^{-1} + HX(H^{-1})'$, and $(H^{-1})' = -H^{-1}H'H^{-1}$. So the first term on the right hand side above is $H'XH^{-1} + HX'H^{-1} - HXH^{-1}H'H^{-1}$. The first term on the left hand side and the second on the expanded right hand side are equal and may be canceled. Then premultiplying both sides by H^{-1} and postmultiplying both sides by H reduces the equation to

$$(H^{-1}H' + H^{-1}QH)X - X(H^{-1}H' + H^{-1}QH) = PX - XP$$

which implies that $H^{-1}H' + H^{-1}QH - P$ is central, hence of the form bI_n.

Conversely, suppose that there is an $H \in GL_n(K)$ such that $P + bI_n = H^{-1}H' + H^{-1}QH$. Let $R = P + bI_n$. Then $\mathcal{D}_P = \mathcal{D}_R$ and $R = H^{-1}H' + H^{-1}QH$. We check that the inner automorphism $T_H : M_n(K) \to M_n(K)$, $T_H(X) = HXH^{-1}$, is a differential isomorphism $(M_n(K), \mathcal{D}_R) \to (M_n(K), \mathcal{D}_Q)$:

$$T_H(\mathcal{D}_R(X)) = HX'H^{-1} + HRXH^{-1} - HXRH^{-1}$$

and

$$\begin{aligned} HRXH^{-1} - HXRH^{-1} &= H(H^{-1}H' + H^{-1}QH)XH^{-1} - HX(H^{-1}H' + H^{-1}QH)H^{-1} \\ &= H'XH^{-1} + QHXH^{-1} - HXH^{-1}H'H^{-1} - HXH^{-1}Q \end{aligned}$$

so that

$$T_H(D_R(X)) = (HXH^{-1})' + Q(HXH^{-1}) - (HXH^{-1})Q = \mathcal{D}_Q(T_H(X))$$

as required. □

As the second part of the above proof shows, as far as the differential matrix algebra goes, the scalar a can essentially be taken to be zero. We record this observation formally as a corollary:

Corollary 1. *Let $(M_n(K), \mathcal{D})$ be a differential Azumaya algebra isomorphic to the trivial algebra. Then there are $Q \in M_n(K)$ and $F \in GL_n(K)$ such that $\mathcal{D} = \mathcal{D}_Q$ and $Q = F^{-1}F'$. Conversely, if $F \in GL_n(K)$ and $Q = F^{-1}F'$ then $(M_n(K), \mathcal{D}_Q)$ is isomorphic to the trivial algebra.*

If $E \supseteq K$ is a differential extension field, and $(A, \mathcal{D})$ is a differential Azumaya K algebra, then $(A \otimes_K E, \mathcal{D} \otimes_K D_E)$, where D_E is the derivation of E and $\mathcal{D} \otimes_K D_E = \mathcal{D} \otimes 1 + 1 \otimes D_E$, is a differential Azumaya E algebra. In case $A = M_n(K)$, we identify $A \otimes_K E$ with $M_n(E)$ as usual. The Proposition (or its Corollary) give us a criterion to tell whether $(M_n(K), \mathcal{D})$ becomes trivial after extending scalars to E. To make this explicit, we turn to Picard–Vessiot extensions.

The matrix equation $Q = F^{-1}F'$ means that F is a solution of the matrix differential equation $Y' = YQ$, and conversely any solution F of the latter satisfies $Q = F^{-1}F'$.

A Picard–Vessiot extension of K for the matrix differential equation $Y' = RY$, $R \in M_n(K)$, is a no new constant differential field extension $E \supseteq K$ such that the equation has an invertible matrix solution (a matrix $F \in GL_n(E)$ such that $F' = RF$, and which is minimal over K with this property (E is differentially generated over K by the entries of F).

Theorem 1. *Let $(M_n(K), \mathcal{D})$ be a differential matrix algebra over K. Then there is a Picard–Vessiot extension $E \supseteq K$ such that $(M_n(K), \mathcal{D}) \otimes_K E$ is trivial.*

Proof. We choose $P \in M_n(K)$ so that $\mathcal{D} = \mathcal{D}_P$. Then we let $E \supseteq K$ be a Picard–Vessiot extension for the equation $Y' = P^tY$. By definition, there is $G \in GL_n(E)$ such that $G' = P^tG$. Let $F = G^t$. Then $F' = FP$, so by Corollary (1) we have that $(M_n(K), \mathcal{D}) \otimes_K E$ is trivial. □

3 Automorphisms of matrix differential algebras

Suppose that $(M_n(K), \mathcal{D})$ is a matrix differential algebra which becomes trivial in the Picard–Vessiot extension $E \supseteq K$. There is an action of $G = G(E/F)$ on $(M_n(K), \mathcal{D}) \otimes_K E$ by $1 \otimes G$, whose fixed ring is $(M_n(K), \mathcal{D})$, and there is an action of G on $(M_n(E), (\quad)')$ by $M_n(G)$ (action on the coordinates) whose fixed ring is $(M_n(K), (\quad)')$. By assumption, the two algebras $(M_n(K), \mathcal{D}) \otimes_K E$

and $(M_n(E), (\quad)')$ are isomorphic, so the actions are, in general, different. We begin by analyzing the second $(M_n(G))$ action on the first algebra.

The algebra $(M_n(K), \mathcal{D}) \otimes_K E$ may be described concretely as follows: choose $P \in M_n(K)$ such that $\mathcal{D} = \mathcal{D}_P$. We have $M_n(K) \subseteq M_n(E)$ and we can regard P as belonging to the latter. Thus

$$(M_n(K), \mathcal{D}) \otimes_K E = (M_n(E), \mathcal{D}_P).$$

If $\sigma \in G$ and $X \in M_n(E)$, we let $\sigma(X)$ denote the action on coordinates $M_n(\sigma)(X)$. Under the above identification, this is the same as the (differential automorphism) action of $1 \otimes \sigma$ on $(M_n(K), \mathcal{D}) \otimes_K E$. Note that $\mathcal{D}_P(\sigma(X)) = \sigma(X)' + P\sigma(X) - \sigma(X)P$ and $\sigma(\mathcal{D}_P(X)) = \sigma(X') + \sigma(P)\sigma(X) - \sigma(X)\sigma(P)$ are equal, since σ commutes with derivation on E and $P \in M_n(K)$ is fixed by σ.

Moreover, we have, by assumption, that $(M_n(E), \mathcal{D}_P)$ and $(M_n(E), (\quad)')$ are isomorphic, and we may assume that this isomorphism is the inner isomorphism T_F given by $F \in GL_n(E)$ where $F^{-1}F' = P$. Thus the above G action on $(M_n(E), \mathcal{D}_P)$ can be transported to a G action on $(M_n(E), (\quad)')$ via T_F, so that $\sigma \in G$ acts via $T_F M_n(\sigma) T_F^{-1}$. We denote this action by ${}^{\sigma}X$, $X \in M_n(E)$, so ${}^{\sigma}X = F(\sigma(F^{-1}XF))F^{-1}$.

We recall that F satisfies the differential equation $F' = FP$. This implies that, for $\sigma \in G$, $\sigma(F)' = \sigma(F)P$. It follows that $(\sigma(F)F^{-1})' = 0$ so that $\sigma(F) = {}_{\sigma}DF$ with ${}_{\sigma}D \in GL_n(C)$. Thus $F(\sigma(F^{-1}XF))F^{-1}$, which is also $F(\sigma(F^{-1})\sigma(X)\sigma(F))F^{-1}$ is expanded to

$$FF^{-1}{}_{\sigma}D^{-1}\sigma(X){}_{\sigma}DFF^{-1} = {}_{\sigma}D^{-1}\sigma(X){}_{\sigma}D.$$

By construction, $X \mapsto {}^{\sigma}X$ is a differential algebra automorphism of $(M_n(E), (\quad)')$, and the preceding analysis shows that it is given by the composition $\mathrm{Inn}({}_{\sigma}D^{-1}) \circ M_n(\sigma)$ of inner automorphism by ${}_{\sigma}D^{-1}$ following the action of σ on coordinates.

We summarize the above calculations in the following proposition:

Proposition 3. *Let $E \supseteq K$ be a Picard–Vessiot extension with group $G = G(E/K)$ and let $P \in M_n(K)$ and $F \in GL_n(E)$ be such that $F' = FP$. For $\sigma \in G$ let ${}_{\sigma}D \in GL_n(C)$ denote $\sigma(F)F^{-1}$. For $X \in M_n(E)$, let ${}^{\sigma}X = {}_{\sigma}D^{-1}\sigma(X){}_{\sigma}D$. Then $X \mapsto {}^{\sigma}X$ is a differential automorphism of $(M_n(E), (\quad)')$. Moreover, $\sigma \mapsto {}^{\sigma}(\cdot)$ is a representation of G in the group of differential automorphisms of $(M_n(E), (\quad)')$.*

Additionally, the differential isomorphism $T_F : (M_n(E), \mathcal{D}_P) \to (M_n(E), (\quad)')$, $T_F(X) = FXF^{-1}$, is G equivariant, when G acts on the codomain via the above representation and on the domain via the action on coordinates.

Proof. It remains to show that $\sigma \mapsto {}^{\sigma}(\cdot)$ is a representation and that T_F is G equivariant. For the former, the equations $\sigma(F) = {}_{\sigma}DF$ imply that ${}_{\sigma\tau}D = {}_{\tau}D{}_{\sigma}D$, so that $\sigma \mapsto \mathrm{Inn}({}_{\sigma}D^{-1})$ is a group homorphism. Since ${}_{\sigma}D \in GL_n(C)$, inner automorphism by ${}_{\sigma}D^{-1}$ commutes with $M_n(\sigma)$, from which it follows that $\sigma \mapsto \mathrm{Inn}({}_{\sigma}D^{-1}) \circ M_n(\sigma)$ is a group homomorphism.

The final assertion follows from the description of ${}^{\sigma}(\cdot)$ as the composition $T_F M_n(\sigma) T_F^{-1}$, which implies that $T_F\sigma(X) = {}^{\sigma}(T_F(X))$. $\square$

We conclude this section with some observations on the group $\mathcal{A}_n = \mathrm{Aut}^{\mathrm{diff}}(M_n(E), (\quad)')$ of differential automorphisms of the trivial matrix differential algebra over E. Every such automorphism restricts to an automorphism of the center E, and this restriction homomorphism has a right inverse given by $\gamma \mapsto M_n(\gamma)$. The kernel consists of differential automorphisms trivial on the center. Since $(M_n(E), (\quad)')$ is also $(M_n(E), \mathcal{D}_I)$, its E linear differential automorphisms are, by Proposition (2), inner automorphisms T_S where $S^{-1}S' = dI_n$ for some $d \in E$ ($d = 1 + a$ in the notation of the Proposition). So $S' = dS$, and then any non–zero element entry y of S also satisfies $y' = dy$, and it follows that $S = yD$ for some $D \in GL_n(C)$. Since $T_S = T_D$, we see that the group of differential automorphisms of $(M_n(E), (\quad)')$ is $\{T_D \mid D \in GL_n(C)\}$. We recognize this latter group as isomorphic to the projective linear group, and so denote it $PGL_n(C)$.

Thus $\mathcal{A}_n$ is the semi-direct product of this normal subgroup and the group of differential automorphisms of E. (A differential automorphism γ congugates an inner automorphism T_S to $T_{\gamma(S)}$, and γ is trivial on C, so the semidirect product is actually a product. We consider its subgroup $PGL_n(G) \times G(E/K)$; it is in this latter group in which the representation of G of Proposition (3) takes values.

4 Cocycles

In the (semi-)direct product description of the differential automorphisms of $(M_n(E), (\quad)')$ which lie over the automorphims $G(E/K)$ of the center, the representation of Proposition (3) is given as $\sigma \mapsto (T_{{}_\sigma D^{-1}}, \sigma)$. More generally, a homomorphism $G \to PGL_n(C) \times G(E/K)$ such that the second coordinate of the image of σ is σ is given by $\sigma \mapsto (\Phi(\sigma), \sigma)$ where $\Phi : G \to PGL_n(C)$ is a one cocyle (here a homomorphism, since G acts trivially), and conversely. In the case of the algebra $(M_n(E), \mathcal{D}_P)$ of Proposition (3), this cocycle is $\Phi : G \to PGL_n(C)$ by $\sigma \mapsto T_{{}_\sigma D^{-1}}$. We note that this is algebraic: recall that ${}_\sigma D$ is defined by $\sigma(F) = {}_\sigma DF$, and that $F' = FP$. Taking transposes, we have that $\sigma(F^t) = F^t {}_\sigma D^t$ and that $(F^t)' = P^t F^t$. So we recognize $\sigma \mapsto {}_\sigma D^t$ as the representation of $G \to GL_n(C)$ associated to the Picard–Vessiot extension of K generated by the entries of F^t [7, (2), p. 19] and hence algebraic.

We consider the effect of an isomorphism $(M_n(K), \mathcal{D}_P) \to (M_n(K), \mathcal{D}_Q)$ on the above association. By Proposition (2), such an isomorphism is given by conjugation by a matrix H such that $P = H^{-1}H' + H^{-1}QH$, or equivalently $Q = HPH^{-1} - H'H^{-1}$. This implies that

$$\begin{aligned}(FH^{-1})^{-1}(FH^{-1})' &= HF^{-1}(F(H^{-1})' + F'H^{-1}) \\ &= -HH^{-1}H'H^{-1} + HF^{-1}F'H^{-1} = -H'H^{-1} + HPH^{-1} \\ &= Q\end{aligned}$$

so the association $P \to F$ becomes $Q \to FH^{-1}$.

The inclusion of $PGL_n(C)$ in $PGL_n(E)$ is G equivariant, so we have an associated map

$$\mathrm{Hom}(G, PGL_n(C)) = Z^1(G, PGL_n(C)) \to Z^1(G, PGL_n(E))$$

of cocycle sets. (Cocyles in $Z^1(G, PGL_n(E))$ are rational functions Ψ such that $\Psi(\sigma\tau) = \Psi(\sigma)\sigma(\Psi(\tau))$.) In the case of Proposition (3), since ${}_\sigma D^{-1} = F\sigma(F^{-1})$, we have that the cocycle Φ is actually a coboundary given by F^{-1}: we define $B^1(G, PGL_n(E))$ to be the functions $G \to PGL_n(E)$ given by $\sigma \mapsto J\sigma(J)^{-1}$ for some $J \in PGL_n(E)$. (Note that this is the correct side for the G action to agree with our definition of cocycle.) Under the isomorphism $(M_n(K), \mathcal{D}_P) \to (M_n(K), \mathcal{D}_Q)$ given by conjugation by H, the corresponding cocycle is the coboundary given by $(FH^{-1})^{-1}$, or $\sigma \mapsto FH^{-1}\sigma((FH^{-1})^{-1})$. When $H \in GL_n(K)$, so $\sigma(H) = H$, this shows that the coboundaries given by F^{-1} and $(FH^{-1})^{-1}$ are equal.

If the matrix M in $GL_n(E)$ represents $J \in PGL_n(E)$, then so does aM for any non–zero $a \in E$, and $aM\sigma(aM)^{-1} = a\sigma(a)^{-1}M\sigma(M)^{-1}$.

This observation will be relevant when we prove the converse of the above cocyle construction, which is the main result of this section.

Theorem 2. *Let $E \supseteq K$ be a Picard–Vessiot extension with differential Galois group G, and let $PGL_n(C)$ be represented as the group of inner differential automorphisms of $(M_n(E), (\quad)')$. Then there is a one to one correspondence between K isomorphism classes of matrix differential K algebras trivialized by E and homomorphisms $G \to PGL_n(C)$ which, as cocycles in $Z^1(G, PGL_n(E))$ is a coboundaries.*

In particular, if $(M_n(K), \mathcal{D})$ is such a K algebra, with $\mathcal{D} = \mathcal{D}_P$ and $F \in GL_n(E)$ is such that $P = F^{-1}F'$, the corresponding cocycle is $X \mapsto {}^\sigma X$ where ${}^\sigma X = {}_\sigma D^{-1}\sigma(X)_\sigma D$ and ${}_\sigma D \in GL_n(C)$ is $\sigma(F)F^{-1}$; this cocycle is the coboundary associated to F^{-1}.

And if $\Lambda : G \to PGL_n(C)$ is a cocyle which is the coboundary associated to a matrix F^{-1}, for $\sigma \in G$, let $\rho(\sigma) = \Lambda(\sigma)M_n(\sigma)$ be the corresponding differential automorphism of $(M_n(E), (\quad)')$ reducing to σ on the center. Then $M_n(E)^{\rho(G)} = FM_n(K)F^{-1}$. Let $Q = F^{-1}F'$. Then $M_n(K)$ is $\mathcal{D}_Q$ stable, and

$$T_{F^{-1}} : (M_n(K), \mathcal{D}_Q) \to (M_n(E)^{\rho(G)}, (\quad)')$$

is an isomorphism, $(M_n(K), \mathcal{D}_Q)$ is trivialized by E, and Λ is the associated cocycle.

Proof. The main assertion of the theorem follows from the two particular assertions. The first one is established in Proposition (3). For the second, we introduce the following notation: let $\Lambda(\sigma)$ be given by inner automorphism by $C(\sigma) \in GL_n(C)$. Then Λ being the coboundary associated to F^{-1} means that $F^{-1}\sigma(F) = d_\sigma C(\sigma)$ for some $d_\sigma \in E$. We usually write this as $\sigma(F) = d_\sigma FC(\sigma)$. Since

$$\rho(\sigma)(X) = \Lambda(\sigma)(\sigma(X)) = C(\sigma)\sigma(X)C(\sigma)^{-1},$$

we have

$$FC(\sigma)\sigma(X)C(\sigma)^{-1}F^{-1} = \sigma(FXF^{-1}), \text{ or}$$

$$T_F(\rho(\sigma)(X)) = \sigma(T_F(X)).$$

Thus X is $\rho(G)$ invariant if and only if $T_F(X)$ is G invariant (under the coordinate action). Since the G invariants of $M_n(E)$ under the coordinate action is $M_n(K)$, we have that $F^{-1}M_n(K)F = M_n(E)^{\rho(G)}$. Let $S = F^{-1}$ and consider the isomorphism $T_S : M_n(K) \to M_n(E)^{\rho(G)}$. Let $\mathcal{D}_0$ denote $(\quad)'$ on $M_n(E)$ restricted to $M_n(E)^{\rho(G)}$. By transport of structure, we define a derivation $\mathcal{D}$ on $M_n(K)$ by $\mathcal{D}(X) = T_S \circ \mathcal{D}_0 \circ {T_S}^{-1}$. Then $\mathcal{D}(X) = F^{-1}(FXF^{-1})'F$. Expanding and simplifying, we have that $\mathcal{D}(X) = X' + F^{-1}F'X - XF^{-1}F'$. Thus $\mathcal{D} = \mathcal{D}_Q$ where $Q = F^{-1}F'$. (We note here that Q is not necessarily in $M_n(K)$, although we must have $Q = aI_n + P$ for some $P \in M_n(K)$ and $a \in E$.) By construction,

$$T_S : (M_n(K), \mathcal{D}_Q) \to (M_n(E)^{\rho(G)}, \mathcal{D}_0)$$

is an isomorphism. That $(M_n(K), \mathcal{D}_Q)$ is trivialized by extending scalars to E follows either from Proposition (2) or directly from the observation that $M_n(E)^{\rho(G)}E = M_n(E)$. Finally, the associated cocycle is produced from S by the equations $\sigma(S) = {}_\sigma DS$. Also, we have $\sigma(F) = d_\sigma FC(\sigma)$. Since $S = F^{-1}$, we conclude that $C(\sigma) = d_\sigma^{-1}{}_\sigma D$, and hence that $C(\sigma)$ and ${}_\sigma D^{-1}$ produce the same inner automorphism, and hence the that the associated cocycle is Λ. $\square$

5 Example: $K = C$

As we noted in the introduction, there can be non–trivial differential Azumaya algebras even in the case that K is algebraically closed. We consider now the case where $K = C$ is algebraically closed and has trivial derivation. In this case, by Proposition (2), $(M_n(C), \mathcal{D}_P)$ is isomorphic to $(M_n(C), \mathcal{D}_Q)$ if and only if there is $H \in GL_n(C)$ and $b \in C$ such that $H^{-1}QH = P + bI_n$. If we let $VM_n(C)$ denote $M_n(C)/KI_n$, and let $GL_n(C)$ act on $VM_n(C)$ by conjugation, then isomorphism classes of differential matrix algebras correspond to $GL_n(C)$ orbits in $VM_n(C)$. These orbits in turn correspond to C matrices in Jordan canonical form, up to the usual permutation of their Jordan blocks and scalar translation of their eigenvalues.

Note that, in particular, these orbit spaces are infinite. Moreover, we claim that we have an injection

$$VM_n(C)/GL_n(C) \to \mathcal{B}r^{\mathrm{diff}}(K).$$

This follows from stabilization: there is a map $VM_n(C) \times VM_m(C) \to VM_{nm}(C)$ induced from $(P, Q) \mapsto P \otimes I_m + I_n \otimes Q$ corresponding to tensor product of differential matrix algebras. These are compatible with the $GL(C)$ conjugation action. Tensor product with the trivial algebra corresponds to combining with the identity matrix, or $(P, I) \mapsto P \otimes I_m + I_n \otimes I_m$. Since $I_n \otimes I_m$ is scalar, therefore trivial in $V_{nm}(C)$, this amounts to $P \mapsto P \otimes I_m$. As usual, we call this map $V_n(C) \to V_{nm}(C)$ *stabilization.* If $P_1, P_2 \in M_n(C)$ determine the same element of the differential Brauer group, then they become equal after some stabilization, so that $P_1 \otimes I_m$ and $P_2 \otimes I_m$ are conjugate up to scalar translation. We can order bases so that matricially $P_i \otimes I_m$ is in block diagonal form with m diagonal blocks each equal to P_i. If each P_i is in Jordan form, so is $P_i \otimes I_m$, and it follows that the latter being conjugate up to scalar translation means that, up to scalar translation, P_1 and P_2 have the same Jordan blocks with the same mulitplicities, and hence determine the same element of $VM_n(C)$. Hence the injection to the differential Brauer group.

6 Splitting Differential Azumaya Algebras

Let $(A, \mathcal{D})$ be a differential Azumaya algebra over K. There is a finite Galois extension $K_1 \supseteq K$ such that $A \otimes_K K_1$ is a matrix algebra [6, Cor. 7.8, p. 47]. Also, K_1 carries a unique derivation making it a Picard–Vessiot extension of K, so that $(A, \mathcal{D}) \otimes_K K_1$ is a differential matrix algebra over K_1. By Corollary (1), there is a Picard–Vessiot extension $E_1 \supseteq E$ such that this differential matrix algebra is trivial over E_1. We want to show now that E_1 can be embedded in a Picard–Vessiot extension $E \supseteq K$, so that $(A, \mathcal{D}) \otimes_K E$ is a trvial matrix differential algebra.

The algebra $(A, \mathcal{D})$ plays no role in the construction of E: we simply need to know that a Picard–Vessiot extension of a finite Picard–Vessiot extension can be embedded in a Picard–Vessiot extension. This is obvious if both are finite (and false if neither is).

Proposition 4. *A Picard–Vessiot extension of a finite Picard–Vessiot extension can be embedded in a Picard–Vessiot extension.*

Proof. Let $E_1 \supseteq K_1$ and $K_1 \supseteq K$ be Picard–Vessiot extensions, with $K_1 \supseteq K$ finite. We suppose that $E_1 \supseteq K_1$ is Picard–Vessiot for the linear operator $L_1 = X^{(m)} + a_{n-1}X^{(m-1)} + \cdots + a_0 X^{(0)}$ with $a_i \in K_1$. For each $\sigma \in G(K_1/K)$, we let L_σ denote the operator obtained from L by applying σ to the coefficients of L. Let K_2 denote a Picard–Vessiot closure of a Picard–Vessiot closure of K [4, Notation 2, p. 162]. Inside K_2, we may consider a Picard–Vessiot extension E_σ of K_1 for L_σ (when σ is the identity, $E_\sigma = E_1$), and we let M be a compositum of E_σ, $\sigma \in G(K_1/K)$. Then M is a Picard–Vessiot extension of K_1. We are going to show it is a Picard–Vessiot extension of K.

We begin by showing it is normal. Let τ be a differential automorphism of K_2 over K. Then τ stabilizes K_1, so that the restriction $\tau|K_1$ is some $\sigma \in G(K_1/K)$. Let $V_\sigma = L_\sigma^{-1}(0)$. If $v \in V_1$, $L_1(v) = 0$ implies that $\tau(L_1(v)) = L_\sigma(\tau(v)) = 0$, so that $\tau(V_1) = V_\sigma$. This implies that $\tau(E_1) \subseteq M$, and similar reasoning applied to the other solution spaces implies that $\tau(M) = M$.

Since $M \supseteq K$ and $K_1 \supseteq K$ are both normal subextensions of $K_2 \supseteq K$. we have surjections $G(K_2/K) \to G(M/K)$ and $G(K_2/K) \to G(K_1/K)$ [4, Lemma 20, p. 163], which implies that $G(M/K) \to G(K_1/K)$ is surjective as well. Note that the kernel of this latter is $G(M/K_1)$.

Now let $W \subset K_1$ be a finite dimensional C vector space stable under $G(K_1/K)$ and generating K_1 linearly over K, and let $V = W + \sum_\sigma V_\sigma$. Then M is differentially generated over K by V. Any $\tau \in G(M/K)$ stabilizes W and permutes the V_σ's, so that V is $G(M/K)$ stable. Since the fixed field of K_2 under $G(K_2/K)$ is K (this is a consequence of [4, Lemma 20, p. 163]; see [3, p. 17]), that of M under $G(M/K)$ is K, and now [2, Prop. 3.9, p. 27] implies that M is a Picard–Vessiot extension of K. $\square$

As a corollary, we deduce the existence of Picard–Vessiot splitting fields:

Corollary 2. *Let $(A, \mathcal{D})$ be a differential Azumaya algebra over K. Then there is a Picard-Vessiot extension E of K such that $(A, \mathcal{D}) \otimes_K E$ is a trivial matrix differential algebra over E.*

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