Differential Projective Modules and Azumaya Algebras over Differential Rings

Lourdes Juan and Andy Magid

Abstract. Differential modules over a commutative differential ring which are finitely generated and projective as ring modules, with differential homomorphisms, form an additive category. All such are shown to be direct summands of objects which are free as ring modules; those which are differential direct summands of differential direct sums of the ring are shown to be induced from the subring of constants. And any object has this form after a suitable extension of the base. Thus the K theory of the differential category reduces to that of ordinary K theory and kernels. Differential Azumaya algebras over the ring whose underlying modules are finitely generated and projective form a multiplicative category, and similar results to the above are obtained. The K theory of this multiplicative category can accordingly be analyzed in a similar way.

1. Introduction

A differential module over a differential ring is a module equipped with an (abelian group) endomorphism which satisfies a product rule (with respect to the ring derivation) for scalar multiplication. These have been studied in many places: for example [**7**], [**6**], [**1**]. Our intention here is to study the K theory of differential projective modules and differential Azumaya algebras over a commutative differential ring R. Thus we focus on projective modules.

The category of differential modules over R contains free, and hence projective, objects, as we recall below. Except in trivial cases, however, these are never finitely generated as R modules, as we also show. Thus we prefer to focus our attention on differential modules which are finitely generated and projective as R modules, what we term *differential finitely generated projective modules*. Direct sums of copies of the ring R, and their differential direct summands are examples. An important observation [see Theorem 1 below] is that such modules are always obtained by tensoring up to R a finitely generated projective module over the ring R^D of constants of R. Since there usually are differential finitely generated projective modules which are not of this form, this class of modules does not exhaust the subject. We do show that every differential finitely generated projective R module is a differential direct summand of a differential finitely generated projective module which is free as an R module [see Corollary 1]. Thus we can regard differential projective modules as direct summands of differential projectives which are free as

1991 *Mathematics Subject Classification.* 12H05.

modules. Further, we show that given any projective finitely generated differential module R module M, there is a commutative differential ring $S \supseteq R$ such that $S \otimes_R M$ is a differential direct summand of a direct sum of copies of S [see Theorem 2]. Moreover, S can be taken to be a localization of a polynomial ring over R at a single element.

The K group of the category of differential finitely generated projective modules (which is not an abelian category but does have a direct sum operation) we call $K_0^{\mathrm{diff}}(R)$. The subgroup formed by the differential direct summands of direct sums of copies of R coincides (by the observation already noted) with the image of $K_0(R^D)$ under $R \otimes_{R^D} (\cdot)$. And by our result above every element of $K_0^{\mathrm{diff}}(R)$ lies in such an image after a suitable extension from R to S. These two results reduce the study of $K_0^{\mathrm{diff}}(R)$ to the study of kernels of $K_0^{\mathrm{diff}}(R) \to K_0^{\mathrm{diff}}(S)$ where S is an extension of the type above.

By a *differential Azumaya* R algebra A we mean an Azumaya algebra A over R which is also a differential ring such that the derivation of A on R is that of R; that is, the derivation of A makes A a differential R module. To say that A is an Azumaya R algebra means (1) that A is a finitely generated projective R module and (2) that $A \otimes_R A^{\mathrm{op}} \to \mathrm{Hom}_R(A, A)$ is an isomorphism of algebras. Note that when A is a differential R algebra the map in (2) is a morphism of differential R algebras. Also note that the definition only asks that A be projective as an R module, not free as an R module. For the definition of Azumaya algebra, (2) may be replaced by the assertion, which can be shown to be equivalent, $(2')$ There is an R algebra B and a finitely generated projective module M such that $A \otimes_R B \cong \mathrm{End}_R(M)$. This can be further refined to $(2'')$ to require that M be R free. We show that for a differential R algebra A which is a differential finitely generated projective R module that A is differential Azumaya if and only if there is a differential R algebra B and a differential finitely generated projective module M such that $A \otimes_R B \cong \mathrm{End}_R(M)$, and that M can be taken to be free as an R module [see Proposition 3]. Thus we can regard differential Azumaya R algebras as tensor factors of differential algebras which are matrix rings as R algebras. As is the case with modules, given a differential Azumaya R algebra A, we can construct $S \supseteq R$ such that $S \otimes_R A$ is a differential tensor factor of differential matrix algebra over R where the derivation is that of R applied to the matrix entries [see Theorem 3]. Moreover, S can be taken to be the localization of a polynomial ring over R at a single element.

The K group of the category of differential Azumaya algebras (with the tensor product operation) we call $K_0\mathrm{Az}^{\mathrm{diff}}(R)$. The subgroup formed by the differential tensor factors of matrix algebras with entry–wise derivation R coincides (by an observation similar to that for modules) with the image of $K_0\mathrm{Az}(R^D)$ under $R \otimes_{R^D} (\cdot)$. And by our result above every element of $K_0\mathrm{Az}^{\mathrm{diff}}(R)$ lies in such an image after a suitable extension from R to S. These two results reduce the study of $K_0\mathrm{Az}^{\mathrm{diff}}(R)$ to the study of kernels of $K_0\mathrm{Az}^{\mathrm{diff}}(R) \to K_0\mathrm{Az}^{\mathrm{diff}}(S)$ where S is an extension of the type above.

As a general reference to K theory of both projective modules and Azumaya algebras we cite [**2**].

Previous work of the authors [**4**] and the second author [**6**] have stressed the case where R is a field, or more generally a simple differential ring. Under the assumption of simplicity, every R finitely generated differential module is R projective [**1**, Theorem 2.2.1] (or see the exposition in [**6**, Theorem 5]). Here we consider when R is not a simple differential ring. Two important types of examples which we will use include the case that $R = \mathbb{C}[x_1, \ldots, x_n]_f$ is the coordinate ring of an affine open subset of complex affine space, with a suitable derivation, or that $R = \mathcal{O}(\mathbb{C})$ is the ring of entire functions on the complex plane. This later object has some properties which may not be as familiar as the former, so we will observe them below.

2. Projective Modules

Let R be a commutative ring with derivation D. The *ring of twisted differential polynomials* over R, denoted $R[X; D]$ is the R module $R[X]$ (ordinary R polynomials in one variable X) with associative, distributive multiplication determined by the rule $Xa = aX + D(a)$ for $a \in R$. By construction, R is a subring of $R[X; D]$, although R is not central in $R[X; D]$ unless D is trivial ($D = 0$), and hence $R[X; D]$ is not usually an R algebra.

A module over $R[X; D]$ is called a *differential R module*. It is straightforward to verify that a differential R module M is an R module with an additive endomorphism D_M given by $D_M(m) = Xm$ which obeys the formula $D_M(am) = D(a)m + aD_M(m)$ for $a \in R$ and $m \in M$; and conversely. A homomorphism of differential R modules is called a *differential homomorphism*. A differential homomorphism $f : M \to N$ between differential R modules is seen to be an R module homomorphism that satisfies $f(D_M(m)) = D_N(f(m))$ for $m \in M$; and conversely. The category of differential R modules, being the category of modules over the ring $R[X; D]$, is an abelian category. A projective object in this category is, by definition, a projective differential module. A *constant* of a differential ring or module is an element of derivative zero. The constants of R form a subring denoted R^D; the constants of a differential R module M form an R^D submodule of M denoted M^D. For a differential module M, there is a map $R \otimes_{R^D} M^D \to M$. If this map is surjective, we say that M is *constantly generated*. If this map is an isomorphism, we say that M is *induced* [from constants]. In Example 3 we show that constantly generated need not be induced.

We fix the above notations.

A differential module may be finitely generated as an R module. For example, this is true of R itself, using D for the endomorphism D_R. However this is not true for (nonzero) projective differential modules.

Proposition 1. *Let P be a projective differential module finitely generated as an R module. Then $P = 0$*

Proof. Since P is R finitely generated, it is $R[X; D]$ finitely generated, which means that there is a differential surjection $R[X; D]^{(n)} \to P$. Since P is projective, this surjection differentially splits and P can be regarded as a differential submodule of $R[X; D]^{(n)}$. The projections $p_i : R[X; D]^{(n)} \to R[X; D]$ are all differential. If $P \neq 0$ then for some i $p_i(P)$ is a nonzero R finitely generated differential submodule of $R[X; D]$. Suppose $g_1, \ldots, g_k$ generate $p_i(P)$ as an R module. Then any element $g \in p_i(P)$ is of the form $\sum_{i=1}^k r_i g_i$. In particular, its degree in X is bounded. Let $f \in p_i(P)$ be a nonzero element with highest degree term $a_m X^m$ where $a_m \neq 0$. Since $X a_m X^m = a_m X^{m+1} + D(a_m)X^m$, the elements $Xf, X^2 f, X^3 f, \ldots$ all lie in

$p_i(P)$ and have strictly increasing degrees. This contradicts boundedness of degrees, and hence we conclude that $P = 0$. $\square$

Proposition 1 shows that there will be no interesting projective differential modules that are finitely generated as R modules. We could go on to consider all differential modules which are finitely generated as R modules; it turns out that the class of interest is differential R modules which are finitely generated and projective as R modules. We single out this terminology, which we have already been using, with a formal defintion.

DEFINITION 1. *A differential R module which is finitely generated and projective as an R module is said to be* differential finitely generated projective.

Next, we see that every finitely generated projective R module is the underlying module of a differential finitely generated projective module.

PROPOSITION 2. *Let M be a projective R module. Then there is an additive endomorphism $D_M : M \to M$ that makes M a differential R module.*

PROOF. Let $R[\varepsilon]$ denote the ring of dual numbers over R (The quotient $R[t]/Rt^2$ of the polynomial ring with $\varepsilon = t + Rt^2$). The derivation D of R defines a ring homomorphism $\Phi : R \to R[\varepsilon]$ by the formula $r \mapsto r + D(r)\varepsilon$. Note that Φ makes $R[\varepsilon]$ an R algebra via Φ. Consider the $R[\varepsilon]$ module $R[\varepsilon] \otimes_\Phi M$, which we denote $M[\varepsilon]$, which is an R module via Φ. The projection $\Psi_M : M[\varepsilon] \to M$ by $m+n\varepsilon \mapsto m$ satisfies $\Phi(r)(m + n\epsilon) = (r + D(r)\varepsilon)(m + n\varepsilon) = rm + (rn + D(r)m)\varepsilon \mapsto rm$ and hence is R linear. As M is a projective R module, there is an R module homomorphism $\psi_M : M \to M[\varepsilon]$ such that $\Psi_M(\psi_M(m)) = m$ for all $m \in M$. Note that if $\psi_M(m) = a + b\varepsilon$ then $m = \Psi_M(\psi_M(m)) = \Psi_M(a + b\varepsilon) = a$. We define D_M by $\psi_M(m) = m + D_M(m)\varepsilon$. Since ψ_M is an R homomorphism, $\psi_M(rm) = \Phi(r)\psi_M(m)$, so $rm + D_M(rm)\varepsilon = (r + D(r)\varepsilon)(m + D_M(m)\varepsilon)$ which implies that $D_M(rm) = D(r)m + rD_M(m)$. Similarly, since $\psi_M(m + n) = \psi_M(m) + \psi_M(n)$, $D_M(m + n) = D_M(m) + D_M(n)$. Thus D_M is the desired additive endomorphism. $\square$

We note that if the projective module M in Proposition 2 is finitely generated then D_M makes M a differential finitely generated projective module.

Proposition 2 implies that the additive and multiplicative trivialization theorems for projective modules apply to differential projective modules.

COROLLARY 1. *Let M be a differential finitely generated projective R module. Then there is a differential finitely generated projective R module N such that $M \oplus N$ is a free R module of finite rank.*

PROOF. There is a finitely generated projective R module N such that the R module $M \oplus N$ is free of finite rank. By Proposition 2, there is a D_N that makes N a differential module. Then $M \oplus N$ is a differential module using the differential structures of M and N. $\square$

COROLLARY 2. *Let P be a differential finitely generated projective R module, and assume P is faithfully projective as an R module.. Then there is a differential finitely generated projective R module Q such that $P \otimes_R Q$ is a free R module of finite rank.*

Proof. By Bass's Theorem [**3**, Proposition 4.6, p 476], there is a finitely generated projective R module Q such that the R module $P \otimes_R Q$ is free of finite rank. By Proposition 2, there is a D_Q that makes Q a differential module. Then $P \otimes_R Q$ is a differential module using the differential structures of P and Q. $\square$

Corollary 1 implies that every differential finitely generated projective module is a direct summand of one which is free an an R module. Among the latter are the direct sums of the differential module R. As we now see, these modules form a distinct, and usually proper, class.

Theorem 1. *Let M be a differential finitely generated projective R module. Then M is a direct summand of a finite number of copies of the differential module R if and only if there exists a finitely generated projective R^D module M_0 and a differential isomorphism $R \otimes_{R^D} M_0 \cong M$.*

Proof. Let M and N be differential modules with $M \oplus N \cong R^{(n)}$. Passing to constants, we have $M^D \oplus N^D \cong (R^D)^{(n)}$ which implies that M^D is a finitely generated projective R^D module. There are differential maps $R \otimes_{R^D} M^D \to M$ and $R \otimes_{R^D} N^D \to N$. Their direct sum is a map $(R \otimes_{R^D} M^D) \oplus (R \otimes_{R^D} N^D) \to M \oplus N$ which is the isomorphism $R \otimes_{R^D} (R^D)^{(n)} \to R^{(n)}$. It follows that both the summand maps are isomorphisms, and in particular$R \otimes_{R^D} M^D \to M$ is an isomorphism.

Conversely, if M_0 is a finitely generated projective R^D module, then there is a finitely generated projective R^D module N_0 such that $M_0 \oplus N_0 \cong (R^D)^{(m)}$ for some m. Then tensoring with R over R^D shows that $(R \otimes_{R^D} M_0) \oplus (R \otimes_{R^D} N_0) \cong r \otimes_{R^D} (R^D)^{(n)} = R^{(n)}$. $\square$

Theorem 1 suggests that we consider the functor $R \otimes_{R^D} (\cdot)$, which takes finitely generated projective R^D modules to differential finitely generated projective R modules: Theorem 1 says what the image is on objects. For later use, we record the following property of this functor.

Lemma 1. *Let M_0 be a finitely generated projective R^D module. Then $M_0 \to (R \otimes_{R^D} M_0)^D$ by $m \mapsto 1 \otimes m$ is a bijection.*

Proof. The map $M_0 \to (R \otimes_{R^D} M_0)^D$ is natural in M_0 and additive. So, as usual with finitely generated projective modules, it suffices to prove bijection for the case $M_0 = R^D$, where it is trivial since $R \otimes_{R^D} R^D = R$. $\square$

In the special case that all differential finitely generated projective R modules are direct sums of copies of R differentially, Theorem 1 shows that all such modules are induced.

Corollary 3. *Assume that all differential finitely generated projective R modules are direct sums of copies of R. Then all differential finitely generated projective R modules are of the form $R \otimes_{R^D} M_0$ where M_0 is a finitely generated projective R^D module.*

Theorem 1 can be used to construct examples of differential finitely generated projective modules. If T is any commutative ring , and $R = T[z, z^{-1}]$ is the ring of Laurent polynomials over T, then R is a differential ring with derivation determined by $D(z) = z$ and $D(T) = 0$. There is a ring homorphism $R \to T$ given by $z \mapsto 1$. If T is an integral domain of characteristic 0, then $R^D = T$. Thus differential finitely generated projective R modules which are direct summands of finitely many copies of R are all of the form $R \otimes_T P$ for some finitely generated projective T module P. By varying T we can obtain examples of various types.

Corollary 1 shows that every differential finitely projective module M appears as a summand of a differential finitely generated projective module which is free as an R module. If this latter module is constantly generated, then Theorem 1 shows that M is induced from R^D. Of course, the module may not be constantly generated. However, as we now show, it is constantly generated after a faithfully flat base change.

To begin, let P be a differential finitely generated projective module which is free as an R module, and suppose that $\{x_1, \dots, x_n\}$ is an R module basis of P. If $m = \sum r_i x_i$ is an element of M, then $D(m) = \sum D(r_i)x_i + r_i D(x_i)$. Suppose that $D(x_i) = \sum a_{ij} x_j$ for $1 \leq i \leq n$. Then $D(m) = \sum b_i x_i$ where $(b_1, \dots, b_n) = (D(r_1), \dots D(r_n)) + (r_1, \dots, r_n)A$, where A is the $n \times n$ matrix with i, j entry $a_{i,j}$. If we dispense with the basis and simply identify P with the set of n tuples of elements of R, then D_M applied to an n tuple $\underline{m}$ is $(\underline{m})' + \underline{m}A$, where $(\cdot)'$ applied to a tuple (or matrix) means to apply D to each entry. If Y is any $n \times n$ matrix then $D(\underline{m}Y) = (\underline{m}Y)' + \underline{m}YA$, while $(\underline{m}Y)' = \underline{m}'Y + \underline{m}Y'$. If we apply this where $\underline{m}$ ranges over the standard basis tuples, then the rows of Y are constants provided $Y' + YA$ is the zero matrix. There may not be such a matrix Y over R. However we can always adjoin elements to R to obtain such a matrix: let z_{ij}, $1 \leq i, j \leq n$ be inderminates over R and form the polynomial ring $R[z_{ij}] := R[z_{1,1}, \dots z_{n,n}]$. Define a derivation on this polynomial ring so that if Z is the $n \times n$ matrix over it with i, j entry z_{ij} then $Z' = -ZA$. By the above, the rows of Z are constants in $R[z_{ij}] \otimes_R P$. If we further make Z be invertible by localizing $R[z_{ij}]$ at the determinant $d = \det(Z)$ then the rows of Z become a basis of constants of $R[z_{ij}][d^{-1}] \otimes_R P$. (This construction is the same as the first steps of the construction of the Picard–Vessiot ring extension for the module P; see [**7**] and [**5**].)

For future reference, we denote the free R module of rank n with the derivation $\underline{m} \mapsto (\underline{m})' + \underline{m}A$, where A is any matrix in $M_n(R)$ as $P(A)$ and we denote the differential R algebra $R[z_{ij}][d^{-1}]$ with derivation determined by $Z' = -ZA$ by $S(A)$. Note that $S(A)$ is faithfully flat as an R algebra: in fact there is an R algebra augmentation $\epsilon : S(A) \to R$ determined by $Z \mapsto I_n$ (which is not a differential augmentation, of course). And if R happens to be the coordinate ring of an affine open subset of affine space, so is $S(A)$, although the ambient affine spaces are not the same.

With these notations, we then have the following theorem:

Theorem 2. *Let M be a differential finitely generated projective R module. Then there is a differential R algebra S, finitely generated, flat, and augmented as an R algebra, such that $S \otimes_R M \cong S \otimes_{S^D} M_0$ where M_0 is a finitely generated projective S^D module.*

Proof. By Corollary 1 M is a differential direct summand of a differential module P which is finitely generated and free as an R module, say of rank n. If a basis is chosen for P, then there is a matrix $A \in M_n(R)$ such that $P \cong P(A)$. Thus M can be considered as a differential direct summand of $P(A)$. Let $S = S(A)$. Since $S \otimes_R P(A)$ is a direct sum of copies of S as a differential module, and $S \otimes_R M$ is a direct summand of $S \otimes_R P(A)$, by Theorem 1 $S \otimes_R M \cong S \otimes_{S^D} M_0$ for some finitely generated projective S^D module M_0. □

Because, in the notation of Theorem 2, S is faithfully flat over R, M can be recovered from $S \otimes_R M$ plus the appropriate descent data. This applies to M as a differential module as the standard descent data is differential. Once we have passed to S, then the extension of M becomes induced (tensored–up) from the constants of S. Thus differential finitely generated projective R modules are obtained from descent of induced–from–constants modules over differential extensions of R which are faithfully flat finitely generated augmented R algebras.

3. Azumaya Algebras

There are a number of equivalent formulations of the definition for an Azumaya algebra over a commutative ring. For our purposes here we adopt the following: an algebra A over the commutative ring R is an *Azumaya algebra* provided that (1) A is a finitely generated projective R module and (2) the map $A \otimes_R A^{op} \to \mathrm{End}_R(A)$, which sends $a \otimes b$ to $x \mapsto axb$, is a bijection. Here A^{op} means the algebra with the same R module structure as A but multiplication with order interchanged: in A^{op}, the product of a and b, in that order, is ba. When needed, we will also write the multiplication on A^{op} as m^{op}, so that $m^{op}(a, b) = ba$.

If R is a differential ring and A is a differential R algebra which is Azumaya as an R algebra, we call A a *differential Azumaya algebra*. This entails A being a differential finitely generated projective module. Let us consider how D_A acts on A^{op}: $D_A(m^{op}(a, b)) = D_A(ba) = D_A(b)a + bD_A(a) = m^{op}(a, D_A(b)) + m^{op}(D_A(a), b)$, which shows that D_A is also a derivation of A^{op}. Thus $A \otimes_R A^{op}$ is also a differential R algebra, as is $\mathrm{End}_R(A)$, using the differential module structure on A, and the map $A \otimes_R A^{op} \to \mathrm{End}_R(A)$ is a differential algebra isomorphism.

Corollary 1 shows that every differential finitely generated projective R module is a direct summand of a differential R module which is free of finite rank as an R module. We next see that a differential Azumaya algebra is a tensor factor of a differential R algebra which is a matrix ring of finite rank as an R algebra. It is a technical requirement of the proof that the Azumaya algebra be faithfully projective as an R module. For convenience in stating the result and subsequent ones we add this mild condition to the definition of Azumaya algebra about. That is, we replace condition (1) by (1)$'$ A is a finitely generated faithfully projective R module.

Proposition 3. *Let A be a differential Azumaya algebra. Then there is a differential Azumaya algebra B such that $A \otimes_R B$ is isomorphic as an R algebra to a matrix ring of finite rank over R.*

Proof. By definition, $A \otimes_R A^{op}$ is differentially isomorphic to $\mathrm{End}_R(A)$. Since A is faithfully projective as well as a differential finitely generated projective module, by Corollary 2 there is a diferential finitely generated projective R module Q such that the differential module $A \otimes_R Q$ is, as an R module, free of finite rank. Note that since Q is a differential module, $\mathrm{End}_R(Q)$ is a differential R algebra, in fact a differential Azumaya algebra. Let $B = A^{op} \otimes_R \mathrm{End}_R(Q)$, which is a differential algebra and Azumaya, being the tensor product of Azumaya algebras. Then $A \otimes_R B \cong \mathrm{End}_R(A) \otimes_R \mathrm{End}_R(Q) \cong \mathrm{End}_R(A \otimes_R Q)$, and since $A \otimes_R Q$ is free as an R module $\mathrm{End}_R(A \otimes_R Q)$ is isomorphic, as an R algebra, to a matrix ring of finite rank over R. □

For later reference, we note that in the proof of Proposition 3 the differential R module $A \otimes_R Q$, which is free of finite rank as an R module, is isomorphic to the differential module $P(C)$ for a suitable matrix C.

All differential finitely generated projective modules were shown above to be direct summands of differential modules which, as R modules, were free of finite rank (Corollary 1). In Theorem 1 we found a criterion for differential finitely generated projective modules to be direct summands of a direct sum of copies of R. We have a related result for Azumaya algebras. A direct sum of copies of R, as a differential module, is a module of tuples where the derivation is the ring derivation applied to each entry. For algebras, the corresponding object is a matrix algebra where the derivation is the ring derivation applied to each matrix entry. In previous notation, this is the differential algebra $\mathrm{End}_R(P(\underline{0}))$, but we will instead just describe it as a matrix ring with entry–wise derivation. In the course of the proof, we will need the following fact about differential matrix algebras:

Lemma 2. *Regard $M_n(R)$ as a differential R algebra via entry–wise derivation, and let B be a differential R subalgebra of $M_n(R)$. Then R is a differential direct summand of B.*

Proof. Let $p_{1,1} : M_n(R) \to R$ be the projection on the $1,1$ entry; this is a differential homomorphism. Let $p : B \to R$ be the restriction of $p_{1,1}$ to B. Since B is an R subalgebra, it contains the identity matrix I_n. Let $q : R \to B$ be given by $q(r) = rI_n$. Because $D(I_n) = 0$, q is also a differential homomorphism. Since $pq = \mathrm{id}_R$, R is a differential direct summand of B. □

Theorem 3. *Let A be a differential Azumaya R algebra. Then A is a tensor factor of a matrix ring with entry–wise derivation over R if and only if there exists an Azumaya R^D algebra A_0 and a differential algebra isomorphism $R \otimes_{R^D} A_0 \cong A$.*

Proof. First assume that there is an R algebra B such that $A \otimes_R B \cong M_n(R)$. We regard both A and B as (differential) subalgebras of $M_n(R)$. By Lemma 2, R is a direct summand of B as a differential R module, say $B = R \oplus M$. Then $A \otimes_R B \cong A \oplus (A \otimes_R M)$, which shows that A is a differential direct summand of $M_n(R)$. Since, as a differential R module, $M_n(R)$ is a direct sum of copies of R, we can apply Theorem 1 to conclude that $R \otimes_{R^D} A^D \to A$ is an isomorphism (of differential modules, but since A^D is a subalgebra of $M_N(R^D)$ and the map $R \otimes_{R^D} A^D \to A$ is an algebra homomorphism, isomorphic as algebras). Similarly, we can apply Lemma 2 to A and conclude that B is a differential direct summand of $M_n(R)$ and hence that $R \otimes_{R^D} B^D \to B$ is an isomorphism. Both A^D and B^D are subalgebras of $M_n(R^D)$, and multiplication defines a map $m : A^D \otimes_{R^D} B^D \to M_n(R^D)$. Thus $M_n(R) \cong A \otimes_R B \cong (R \otimes_{R^D} A^D) \otimes_R (R \otimes_{R^D} B^D)$ which implies that $R \otimes_{R^D} (A^D \otimes_{R^D} B^D) \cong M_n(R)$. Since $M_n(R)^D = M_n(R^D)$ and, by Lemma 1 $(R \otimes_{R^D} (A^D \otimes_{R^D} B^D)^D \cong A^D \otimes_{R^D} B^D)$, we conclude that m is an isomorphism. This imples that A^D (and B^D) is an Azumaya R^D algebra.

Conversely, if A_0 is an Azumaya R^D algebra, then there is an Azumaya R^D algebra B_0 such that $A_0 \otimes_{R^D} B_0 \cong M_n(R^D)$ for some n. (We recalled a proof of this fact in the proof of Proposition 3.) Tensoring this isomorphism with R over R^D gives $(R \otimes_{R^D} A_0) \otimes_R (R \otimes_{R^D} B_0) \cong M_n(R)$, showing that $R \otimes_{R^D} A_0$ is a tensor factor of a matrix ring with entry–wise derivation over R. □

Consider the special case that all differential finitely generated projective R modules are direct sums of copies of R differentially, and let A be a differential Azumaya R algebra. Then as a differential module A is a direct sum of copies of R differentially. If there are n such copies, then $\mathrm{End}_R(A)$ is isomorphic to $M_n(R)$ with entry-wise derivation. Then we can apply Theorem 3 to conclude that A is induced.

Corollary 4. *Assume that all differential finitely generated projective R modules are direct sums of copies of R. Then all differential Azumaya R algebras are of the form $R \otimes_{R^D} A_0$ where A_0 is an Azumaya R^D algebra.*

Proposition 3 shows that every differential Azumaya algebra A appears as a factor of a differential Azumaya algebra which is a matrix ring as an R algebra. If this latter algebra is differential by entry–wise derivation, then Theorem 3 shows that A is induced from R^D. Of course, the algebra may not be differential by entry–wise derivation. However, as we now show, it is differential by entry–wise derivation after a faithfully flat base change.

Thus let B be a differential R algebra whose underlying R algbera is a matrix ring $M_n(R)$. In addition to the derivation D_B, we also have the entry–wise derivation on $M_n(R)$, which we write as Δ_n. Then the difference $D_B - \Delta_n$ is also a derivation of $M_n(R)$, and this one is zero on R. It follows that $D_B - \Delta_n$ is inner and hence given by a matrix $A \in M_n(R)$, so that $D_B(X) = \Delta_n(X) + AX - XA$. (This repeats an argument from [**4**].) To conform with previous notation, we will also write X' for $\Delta_B(X)$. We consider whether $M_n(R)$ with derivation D_B is actually the same as $M_n(R)$ with derivation Δ_n. That is, we ask for an R algebra automorphism σ of $M_n(R)$ such that $\sigma(D_B(X)) = (\sigma(X))'$ for $X \in M_n(R)$.Unlike the field case considered in [**4**], σ need not be inner. But if it is, say if $\sigma(X) = ZXZ^{-1}$ for some invertible Z then calculation shows that $Z^{-1}Z' = A + rI$ for some $r \in R$, and this necessary condition is also sufficient. Taking $r = 0$ and multiplying by Z yields the formula $Z' = ZA$. While there may not be such a Z in $M_n(R)$, there is one in the differential ring $S(-A)$ considered above. And in analogy with the results for differential projective modules, we have the following results for differential Azumaya algebras:

Theorem 4. *Let A be a differential Azumaya algebra over R. Then there is a differential R algebra S, finitely generated, flat, and augmented as an R algebra, such that $S \otimes_R A \cong S \otimes_{S^D} A_0$ where A_0 is an Azumaya S^D algebra.*

Proof. By Proposition 3 A is a differential tensor factor of a differential algebra B which is isomorphic as an R algebra to a matrix ring $M_n(R)$. The derivation D_B has the form Δ_n plus innner derivation by A for some $A \in M_n(R)$.. Let $S = S(-A)$. Since $S \otimes_R B$ is isomorphic to $M_n(S)$ with entry–wise derivation, and $S \otimes_R A$ is a tensor factor of $S \otimes_R B$, by Theorem 3 $S \otimes_R A \cong S \otimes_{S^D} A_0$ for some Azumaya S^D algebra A_0. □

Because, in the notation of Theorem 4, S is faithfully flat over R, A can be recovered from $S \otimes_R A$ plus the appropriate descent data. This applies to A as a differential algebra as the standard descent data is differential. Once we have passed to S, then the extension of A becomes induced (tensored–up) from the constants of S. Thus differential finitely Azumaya R algebras are obtained from descent of induced–from–constants algebras over differential extensions of R which are faithfully flat finitely generated augmented R algebras.

4. K–Theory

The class of differential finitely generated projective R modules is closed under (R module) direct sum. Direct sum can be used to make the set of isomorphism classes of differential finitely generated projective modules into a monoid: If $[\cdot]$ denotes differential isomorphism class then $[M] + [N] := [M \oplus N]$. This is an associative and commutative operation, with identity $[0]$. Following the usual conventions, we denote the most general group to which this monoid maps $K_0^{\mathrm{diff}}(R)$, which we call the *differential K group* of differential projective modules. We denote the image of the isomorphism class $[M]$ in $K_0^{\mathrm{diff}}(R)$ by the same symbol, $[M]$. If S is a differential R algebra there is a homomorphism $K_0^{\mathrm{diff}}(R) \to K_0^{\mathrm{diff}}(S)$ induced from tensoring over R with S. Any element of $K_0^{\mathrm{diff}}(R)$ can be written $[M] - [N]$, where M and N are differential finitely generated projective modules. If P and Q are differential finitely generated projective modules then $[P] = [Q]$ in $K_0^{\mathrm{diff}}(R)$ if and only if there is a differential finite generated projective module M such that $P \oplus M \cong Q \oplus M$.

The group $K_0^{\mathrm{diff}}(R)$ is related to the usual K–theory of R and R^D. Regarding the former, there is a homomorphism $K_0^{\mathrm{diff}}(R) \to K_0(R)$ which, by Proposition 2, is surjective. An element $[M] - [N]$ of $K_0^{\mathrm{diff}}(R)$ lies in the kernel provided that, as R modules, M and N are stably isomorphic. Regarding the latter, there is a homomorphism $K_0(R^D) \to K_0^{\mathrm{diff}}(R)$ given by tensoring over R^D with R. By Theorem 1 the image of this homomorphism is given by (differences of) objects which occur as summands of $m[R]$ for $m = 1, 2, 3, \ldots$. Moreover, there may be a kernel, ultimately due to the fact that a surjective differential homomorphism need not be surjective on constants.

Theorem 2 provides the following theoretical calculation tool for differential K theory of projective modules:

Corollary 5. *Let $x \in K_0^{diff}(R)$. Then there is a differential R algebra, faithfully flat, finitely generated, and augmented as an R algebra such that under $K_0^{diff}(R) \to K_0^{diff}(S)$ the image of x lies in the image of $K_0(S^D) \to K_0^{diff}(S)$.*

Proof. Let $x = [M_1] - [M_2]$ where M_i is a differential finitely generated projective module. Let S_i be the algebra provided for M_i by Theorem 2, and let $S = S_1 \otimes S_2$. Then both M_1 and M_2 are differential direct summands of direct sums of copies of S and hence induced from S^D by Theorem 1. $\square$

We have an analogous situation regarding differential Azumaya algebras. We have previously defined K groups for Azumaya algebras when R was a field [**4**]. The definition below generalizes and supplants our previous definition.

The reader will note that the consideration of differential Azumaya algebras exactly mimics the consideration of differential projective modules that began this section.

The class of differential Azumaya R algebras is closed under (R module) tensor product. Tensor product can be used used to make the set of isomorphism classes of differential Azumaya algebrass into a monoid: If $[\cdot]$ denotes differential isomorphism class then $[M][N] := [M \otimes N]$. This is an associative and commutative operation, with identity $[R]$. Following the notation of [**4**], we denote the most general group to which this monoid maps $K_0^{\mathrm{diff}}\mathcal{A}z(R)$, which we call the *differential K group* of differential Azumaya algebras. We denote the image of the isomorphism class $[A]$ in $K_0^{\mathrm{diff}}\mathcal{A}z(R)$ by the same symbol, $[A]$. If S is a differential R algebra there is a homomorphism $K_0^{\mathrm{diff}}\mathcal{A}z(R) \to K_0^{\mathrm{diff}}\mathcal{A}z(S)$ induced from tensoring over R with S. Any element of $K_0^{\mathrm{diff}}\mathcal{A}z(R)$ can be written $[A][B]^{-1}$, where A and B are differential Azumaya algebras. If A and B are differential Azumaya algebras then $[A]=[B]$ in $K_0^{\mathrm{diff}}\mathcal{A}z$ if and only if there is an Azumaya algebra C such that $A \otimes C \cong A \otimes C$.

The group $K_0^{\mathrm{diff}}\mathcal{A}z(R)$ is related to the usual Azumaya algebra K–theory of R and R^D. Regarding the former, there is a homomorphism $K_0^{\mathrm{diff}}\mathcal{A}z(R) \to K_0\mathcal{A}z(R)$ which, as previously remarked, is surjective. An element $[A][B]^{-1}$ of $K_0^{\mathrm{diff}}\mathcal{A}z(R)$ lies in the kernel provided that, as R algebras, A and B are stably isomorphic. Regarding the latter, there is a homomorphism $K_0\mathcal{A}z(R^D) \to K_0^{\mathrm{diff}}\mathcal{A}z(R)$ given by tensoring over R^D with R. By Theorem 3 the image of this homomorphism is given by (ratios of) objects which occur as factors of $[M_m(R)]$ for $m = 1, 2, 3, \dots$. Moreover, there may be a kernel, ultimately due to the fact that a surjective differential homomorphism need not be surjective on constants.

Theorem 4 provides the following theoretical calculation tool for differential K theory of Azumaya algebras:

Corollary 6. *Let $x \in K_0^{\mathit{diff}}\mathcal{A}z(R)$. Then there is a differential R algebra, faithfully flat, finitely generated, and augmented as an R algebra such that under $K_0^{\mathit{diff}}\mathcal{A}z(R) \to K_0^{\mathit{diff}}\mathcal{A}z(S)$ the image of x lies in the image of $K_0\mathcal{A}z(S^D) \to K_0^{\mathit{diff}}\mathcal{A}z(S)$.*

Proof. Let $x = [A_1][A_2]^{-1}$ where A_i is a differentialAzumaya algebra. Let S_i be the algebra provided for A_i by Theorem 4, and let $S = S_1 \otimes S_2$. Then both A_1 and A_2 are differential factors of matrix algebras over S with entry–wise derivation, and hence induced from S^D by Theorem 3. $\square$

5. Examples

This section collects some examples illustrating some extreme cases of differential rings. We intend to pursue investigations of K-theory calculations in future work.

Example 1. *Let $R = \mathbb{C}$ with $D = 0$.*

In this case all finitely generated R modules are projective, indeed free, and for a differential R module M D_M is R linear. Thus differential finitely generated projective R modules are finite dimensional vector spaces with designated endomorphisms. The structure, classification, and K-theory of these modules is thus the same as that of complex matrices under conjugation. Azumaya algebras over R are all matrix rings, and differential Azumaya algebras are complex matrix rings with a designated inner derivation, and again structure and classification reduces to that of complex matrices under conjugation. We previously discussed this latter in [**4**, Section 5, p.1916-7]

Example 2. *Let $R = \mathcal{O}(\mathbb{C})$ (the ring of entire functions on the complex plane in the variable z) with $D = \frac{d}{dz}$.*

It is a consequence of Wieirstrauss Theory that finitely generated ideals of R are principal. Suppose f is an entire function and that the principal ideal Rf is a differential ideal. Suppose that $f(\alpha) = 0$ for some $\alpha \in \mathbb{C}$, and suppose the order of the zero of f at α is $n > 0$. Then f' has a zero at α of order $n-1$. But since Rf is a differential ideal, f is a factor of f', so that f' has a zero at α of order at least n.

We conclude that f has no zeros, and hence is a unit, so that $Rf = R$ is the only finitely generated differential ideal of R.

On the other hand, R has proper differential ideals: again by Wierstrauss Theory, there is an entire function g which has a zero at n of order n for every natural number $n \in \mathbb{N}$. This means that for any m the functions $g^{(i)}$, $i = 0, 1, \ldots, m$ have infinitely many common zeros at $m+1, m+2, \ldots$ and so the ideal of R generated by $g, g', \ldots, g^{(m)}$ does not contain 1. It follows that the differential ideal I of R generated by g is proper. We note that the differential R module R/I is finitely generated as an R module, but not projective as an R module, since R is an integral domain, and hence its only idempotents are 0 and 1, so the surjection $R \to R/I$ can't split.

Because finitely generated ideals of R are principal, finitely generated projective R modules are free. This means that the differential R modules which are finitely generated and projective as R modules are R free, so have the form $P(A)$ for a suitable matrix A over R. [Monic, homogeneous] linear differential equations with entire coefficients have an entire solution, which implies that $P(A)$ has a basis of constants, and hence is a differential direct sum of copies of R.

By Corollary 3 this means that every differential finitely generated projective R module is induced from $R^D = \mathbb{C}$. (Note: this is $\mathbb{C}$ as a ring, not as a differential ring as in Example 1.) Then Corollary 4 means that every differential Azumaya R algebra is induced from $\mathbb{C}$, and all the latter are matrix rings. Thus the only Azumaya R algebras are matrix rings with entry–wise derivation.

Example 3. *Let R be the localized polynomial ring $\mathbb{C}[a,b,c][h^{-1}]$ where $h = a^2b^2 + a^2c + b^2c$ and $D(a) = a$, $D(b) = b$, $D(c) = c$, and $D(\alpha) = 0$ for $\alpha \in \mathbb{C}$.*

Let A be the 2×2 complex matrix $-I_2$, and consider the differential R module $M = P(A)$, which is free of rank 2 as an R module. For $\underline{x} \in M$, $D_M(\underline{x}) = \underline{x}' + \underline{x}A$. Then $D_M((a,0)) = (0,0)$, $D_M((0,b)) = (0,0)$, and $D_M((c,c)) = (0,0)$, so $(a,0)$, $(0,b)$, and (c,c) are constants. Moreover, $a(a,0) + (c,c) = (a^2+c, c)$ and $b(0,b) + (c,c) = (c, b^2+c)$ are the rows of a 2×2 matrix whose determinant h is a unit of R, and hence they form a basis of M. Thus the constants $(a,0)$, $(0,b)$, and (c,c) span M. Suppose $(x,y) \in M$ is a constant. Then $(0,0) = D_M((x,y)) = (x'-x, y'-y)$ so $D(x) = x$ and $D(y) = y$. Now suppose (x,y) and (z,w) are an R basis of M consisting of constants. The determinant of the matrix with rows (x,y) and (z,w) is $g = xw - yz$ and $D(g) = 2g$. Moreover, g is a unit. Now R is the localization of a unique factorization domain at an irreducible polynomial, so its only units are complex multiples of powers of h. So suppose $g = ch^k$, $\alpha \in \mathbb{C}$, in the polynomial ring. Then $2\alpha^2h^k = 2g = D(g) = \alpha k h^{k-1}D(h)$ which implies that h is a factor of $D(h)$. But $D(h) = 4a^2b^2 + 3b^2c + 3a^2c$ is not a polynomial multiple of h. Thus we conclude that M does not have a basis of constants, despite the fact that it is constantly generated. We now show that this implies M is not induced by showing that $R^D = \mathbb{C}$. Consider an element of R with derivative zero. We can write this as a fraction $\frac{p}{q}$ where p and q are relatively prime polynomials and q is a power of h. Since $\frac{p}{q}$ is a constant, $qD(p) - pD(q) = 0$. If $q \neq 1$ this implies that q is a factor of $D(q)$. Since $q = h^k$ for some $k > 0$, we have h^k as a factor of $kh^{k-1}D(h)$ so that h is a factor of $D(h)$; as noted, this is not possible. Thus $q = 1$ and $D(p) = 0$. Write $p = \sum_{i=0}^n p_i$, where p_i is the term in p of total degree i. Then $D(p) = \sum^n ip_i$ so $D(p) = 0$ implies that $p = p_0 \in \mathbb{C}$.

References

[1] André, Y. "Solution algebras of differential equations and quasi–homogeneous varieties", *Ann. Sci. Éc. Norm. Supér.* **47** (2014), 449-467

[2] Bass, H. and Roy, A. "Lectures on topics in algebraic K-theory. Notes by Amit Roy" *Tata Institute of Fundamental Research Lectures on Mathematics* **41** Tata Institute of Fundamental Research, Bombay 1967

[3] Bass, H. *Algebraic K Theory*, W. A. Benjamin, New York–Amsterdam, 1968

[4] Juan, L. and Magid, A., "Differential central simple algebras and Picard–Vessiot representations", *Proc. Amer. Math. Soc.* **136** (2008), 1911-1918

[5] Magid, A. "Lectures on differential Galois Theory" , *University Lecture Series* **7**, American Mathematical Society, Providence 1994, (second printing with corrections, 1997)

[6] Magid, A. "Differential étale extensions and differential modules over differential rings", *Adv. in Appl. Math.* **72** (2016), 195-214

[7] van der Put, M. and Singer, M. *Differential Galois Theory*, Springer-Verlag, New York, 2003

Department of Mathematics, Texas Tech University, Lubbock TX,
E-mail address: `lourdes.juan@ttu.edu`

Department of Mathematics, University of Oklahoma, Norman OK 73019,
E-mail address: `amagid@ou.edu`